Partial Notes for Calculus I

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2007

1 Polynomial Functions

Mathematics is a language, so we begin with:

1.1 Notation

$p(x)$ refers to a polynomial of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad \text{with } a_i \in \mathbb{R} \text{ and } n \in \mathbb{N}$$

The a_i's refer to fixed real number constants (the notation $a_i \in \mathbb{R}$ means that a_i is an element of the real numbers). The natural number n, ($n \in \mathbb{N}$ means that n is a natural number of the form $1, 2, 3, 4, \ldots$) is called the *degree* of the polynomial, and a_n it's *leading coefficient*. The letter x refers to a variable real number, and can be thought of as the input of the function. The letter p refers to the polynomial function itself, while the notation $p(x)$ refers to the polynomial explicitly as a function of x. If we think of x as the input, then $p(x)$ is the corresponding output. For example, if $p(x) = x^2 - 2$, then for the input $x = 3$ we have the output $p(3) = 3^2 - 2 = 7$. We often use other letters to refer to the variable. For example, for p given above, we could write $p(t) = t^2 - 2$, $p(\gamma) = \gamma^2 - 2$, or $p(\#) = \#^2 - 2$.

1.1.1 Exercise

Let $p(x) = -7x^2 + 4000x^5 - \pi x$. What is the degree of p? What is it's leading coefficient? What is a_1? a_0? What is $p(-1)$? ■

1.2 Roots of Polynomials and Polynomial Equations

The *roots* (or zeroes) of a polynomial p are all values x for which $p(x) = 0$. We often can find roots of polynomials by factoring them.

1.2.1 Exercise

1. Find the roots of the *linear* polynomial $p(x) = 12x - \frac{2}{3}$
2. Find the roots of the *quadratic* polynomial $p(x) = 2x^2 - 6x - 8$

3. Find the roots of the *cubic* polynomial $p(x) = x^3 - x$ ■

The well-known quadratic formula allows us to find the roots of *any* 2^{nd} degree polynomial: If $p(x) = ax^2 + bx + c$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice that this equation identifies two roots of p, although they may not be distinct (i.e. if $b^2 - 4ac = 0$) and they may be purely imaginary (i.e. if $b^2 - 4ac < 0$)

1.2.2 Exercise

Find the roots of the polynomial $p(x) = x^2 + 1$ ■

What about higher degree polynomials? There is no common algorithm like the quadratic formula, and factoring is often difficult. This problem has kept many mathematicians busy throughout history, but today, we can use graphing utility tools to help us. If we look at a graph of p, then the values of x for which $p(x) = 0$ are precisely where the graph touches, or crosses, the x-axis. These are called the *x-intercepts* of the graph.

1.2.3 Exercise

Use your graphing calculator to find the roots of

1. $p(x) = x^4 + 1$
2. $p(x) = x^3 - \frac{1}{4}x^2 - 4x + 1$

■

We can thus use a graphing utility to find all of the real solutions to *any* polynomial equation. We first take the equation and move all of the non-zero terms to one side, giving us an equation of the form $p(x) = 0$. We then find the roots of p to solve the given equation.

Solving the equation *analytically* means factoring p (or using the quadratic formula). If we can't do this, then we solve the equation *graphically* by looking for the x-intercepts of the graph of p. Note however, that graphing utilities will often only give us an *estimate* for the actual solution.

1.2.4 Exercise

Solve the polynomial equation $-2x = 7x^2 - 4$ both analytically and graphically. ■

The *Fundamental Theorem of Algebra* (or FTOA) tells us that any n^{th} degree polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ can (at least theoretically) be factored into the form

$$p(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$$

and thus p has *exactly* n roots and they are the constants $c_1, c_2, \ldots c_n$. However, the roots c_i may not be all distinct, and some may be purely imaginary.

1.2.5 Example

1. The 4^{th} degree polynomial $p(x) = 7x^4 - 7x^3 - 21x^2 + 35x - 14 = 7(x+2)(x-1)^3$. So the 4 roots of p are $x = -2$ and $x = 1$ (where the root $x = 1$ occurs with *multiplicity* 3).

2. The quadratic polynomial $p(x) = x^2 + 1 = (x - i)(x + i)$, so the roots of p are $x = i$ and $x = -i$, where $i = \sqrt{-1}$ is the imaginary unit. ■

1.2.6 Exercise

Find the 3 distinct real roots of the polynomial $p(x) = x^5 - 6x^4 + 12x^3 - 12x^2 + 11x - 6$ graphically. Since p is a 5^{th} degree polynomial, there are supposed to be five roots. What can you say about the other two roots? How might you find them? ■

Another result of the FTOA is that purely imaginary roots always occur in conjugate pairs. That is, if $x = a + bi$ is a root of p, then so is its conjugate, $a - bi$. Thus, the purely imaginary roots can only occur in even numbers.

1.2.7 Exercise

Suppose $1 - i$ is a root of a 7^{th} degree polynomial p. Find another root of p. Find a quadratic polynomial that divides p. What are the possible number of real roots of p? ■

1.3 Polynomial Inequalities

A polynomial inequality is of the form $q(x) \leq r(x)$, where q and r are polynomials and $\leq$ can also be one of $\geq$, $<$, or $>$. As before, we can move all of the non-zero terms to one side, and get an inequality of the form $p(x) \leq 0$.

To solve $p(x) \leq 0$, we first identify all x-intercepts of p (since this is where $p(x) = 0$) and then look between successive x-intercepts to determine if p is positive or negative on that interval. To do this *analytically*, we create a sign chart for p. By identifying all of the real roots on a number line, we can then look at the intervals between successive roots. We choose a test point a from each of these intervals (any convenient point will do) and evaluate p at this point. Since p is always positive or negative on each of these intervals, the sign of $p(a)$ tells us the sign of the polynomial on the entire interval.

1.3.1 Exercise

Use a sign chart to solve the polynomial inequalities

1. $$x^2 - 1 > 0$$

2. $$(2x - 7)(x + 8)(x - 1) \leq 0$$

■

Note that the polynomials in the above exercise are already factored, so determining the roots is immediate. However, we may require a graphing utility to find the roots of p. If this is the case, then a sign chart is redundant, as we have the graph before us and can use it to determine the sign of p on the appropriate intervals.

1.3.2 Example

To solve the polynomial inequality $-15x^2 + 22x < -2x^3 - 15$, we can first rewrite the inequality as $2x^3 - 15x^2 + 22x + 15 < 0$. Next, we graph $p(x) = 2x^3 - 15x^2 + 22x + 15$, and find the roots of p to be $x = -\frac{1}{2},\ 3,\ 5$. Finally, we determine that p lies below the x-axis to the left of the root $x = -\frac{1}{2}$, and between the roots $x = 3$ and $x = 5$. So all values of x in the intervals $(-\infty, -\frac{1}{2})$ and $(3, 5)$ are solutions to the inequality. ■

1.3.3 Exercise

Solve the polynomial inequality $x^6 - 35x^4 + x^5 - 123x^3 \geq 148x^2 + 124x + 112$ graphically. ■

1.4 Homework

1. Solve the following equations and inequalities *analytically*
   (a) $10x + 5 - 7x \leq 8(x + 2) + 4$
   (b) $2x^2 + 4x + 1 = 0$
   (c) $x^3 \geq x$
2. Solve the following equations and inequalities *graphically*
   (a) $3x^4 - 6x^2 = -2x + 5x^3 - 1$
   (b) $3x^4 - 6x^2 < -2x + 5x^3 - 1$
3. Use your graphing calculator to graph $f(x) = x^5 - 4x^4 + 2x^3 - 4x^2 + 6x - 1$ and answer the following:
   (a) Find all real roots of f
   (b) On what intervals is $f(x) > 0$?
   (c) How many imaginary (i.e. non-real) roots does f have?
4. Find a fourth degree polynomial with roots 0 and i. Write your polynomial in standard form $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

5. Consider the polynomial $p(x) = 2x^4 + 2x^3 - 6x^2 - 8x - 8$. Suppose I told you that $x = 2$ and $x = -2$ are roots of p and that the other two roots are imaginary.

   (a) Find a quadratic polynomial that divides p.

   (b) Divide p by your quadratic from part (a) to obtain another quadratic whose roots are the other two roots of p.

   (c) Find all four roots of p

   (d) Write $p(x)$ as a product of linear factors.

2 Rational Functions

2.1 Roots of Rational Functions

A rational function is one of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials. For now, let's assume that p and q have no common root. For if there is some real number a with $p(a) = q(a) = 0$, then $(x - a)$ is a factor of both p and q and we can factor it out and cancel it.

2.1.1 Example

$$\frac{(x-1)^2}{(x-1)(x+2)} = \frac{(x-1)}{(x+2)} \quad \text{(for } x \neq 1\text{)}$$

■

Suppose $p(a) = 0$ and $q(a) \neq 0$. Then $f(a) = \dfrac{p(a)}{q(a)} = \dfrac{0}{q(a)} = 0$. So the roots of f are just the roots of the numerator p.

2.1.2 Exercise

Find (*analytically*) the roots of the rational function

$$f(x) = \frac{x^3 - x^2 - 2x}{2x^5 - 3x^4 + 6}$$

■

2.2 Rational Equations

Equations and inequalities involving rational functions can be treated much in the same manner as polynomials. Remember that when multiplying an inequality by a negative number, one must reverse the inequality. The best way to solve rational equations and inequalities is to first move *all* non-zero terms to one side. Then, using a common denominator, add all of the terms together to get a single rational function.

2.2.1 Example

$$\frac{x-1}{x+1} - \frac{1}{x} + 5 = \frac{x(x-1)}{x(x+1)} - \frac{(x+1)}{x(x+1)} + \frac{5x(x+1)}{x(x+1)}$$

$$= \frac{x(x-1)-(x+1)+5x(x+1)}{x(x+1)} = \frac{x^2-x-x-1+5x^2+5x}{x(x+1)} = \frac{6x^2+3x-1}{x^2+x}$$

■

Once we have this expression set equal to zero, to solve the equation we find the roots of the rational function, which are just the roots of the numerator.

2.2.2 Example

To solve the rational equation

$$\frac{x-1}{x+1} = \frac{1}{x} - 5$$

we first write the equation in the simplified form given in the previous example, and then find the roots of

$$f(x) = \frac{6x^2+3x-1}{x^2+x}$$

The roots of f are the roots of the numerator, $p(x) = 6x^2+3x-1$. Using the quadratic formula, we get that $x = \dfrac{-3 \pm \sqrt{33}}{12}$. It is possible that the solutions we get using this method may make the original expression undefined. For example, if we had gotten $x = 0$ or $x = -1$, these values make a denominator zero and the expression is undefined. Thus, they would not be solutions. It is important to check your solutions in this way. ■

2.2.3 Exercise

Solve the following rational equation *analytically*. Verify with a graphing calculator.

$$\frac{x}{x-2} + \frac{1}{x+2} = \frac{8}{x^2-4}$$

■

2.3 Rational Inequalities

To solve a rational inequality, we first rewrite it as a single rational function on one side of the inequality, and zero on the other (as in the previous 2 examples). We then construct a sign chart for the rational function as we did for polynomial functions. We include the roots of the numerator in our sign chart since a rational function can change sign at a root. However, a rational function can also change sign when the denominator is zero. Thus, we must also include the roots of the denominator in our sign chart (we'll see later that graphically these point correspond to *vertical asymptotes*).

2.3.1 Example

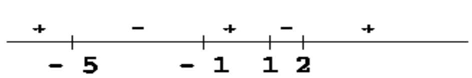
We use a sign chart to solve the inequality analytically.

$$\frac{(x-2)(x+5)}{x^2-1} \geq 0$$

If we let $f(x)$ be the left hand side of the inequality, then the points to include in our sign chart are the roots of the numerator $2, 5$ and roots of the denominator $1, -1$. We then choose test points from each interval and compute the sign of f at each of the test points:

$$f(-6) = \frac{8}{35} \quad f(-3) = -\frac{5}{4} \quad f(0) = 10 \quad f(1.5) = -\frac{13}{5} \quad f(3) = 1$$

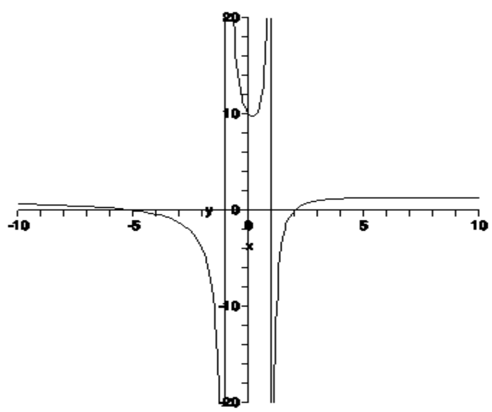
Which gives us the sign chart



Since we want to know where $f \geq 0$, we see that the solution is given by three intervals. Since the inequality is not strict, we include the roots of f, namely the roots of the numerator -5 and 2. However, we do not include the endpoints 1 and -1 since they are roots of the denominator and make f undefined. Thus the solution is

$$(-\infty, -5] \quad (-1, 1) \quad [2, \infty)$$

Finally, we can verify the solution graphically. Notice the *vertical asymptotes* at $x = 1$ and $x = -1$ and that the graph of f is above the x-axis on each of the three intervals given in the solution above.



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2.4 Homework

1. Solve the rational equations analytically.

   (a)
   $$\frac{-1}{(x+3)^2} = 0$$

   (b)
   $$\frac{(x+5)(x-2)}{(x^2-4)} = 0$$

(c)

$$\frac{1}{x} - \frac{x-2}{3-x} = 1$$

2. Using a sign chart, solve the rational inequalities analytically. Verify with a graphing calculator.

   (a)

   $$\frac{-1}{(x+3)^2} > 0$$

   (b)

   $$\frac{x-1}{x+2} > 1$$

3. Solve the inequality graphically.

$$\frac{x^3 - 3x^2 + 7x + 4}{x^4 + 1} \geq 0$$

4. Investigate, graphically, the roots and sign changes of the rational functions. Notice that the numerator and denominator have a common root!

   (a)

   $$f(x) = \frac{x-1}{(x-1)(x+2)}$$

   (b)

   $$f(x) = \frac{(x-1)^2}{(x-1)(x+2)}$$

3 Domain, Range and Composition

3.1 Domain and Range

The *domain* of a function is all possible inputs x. Typically, the domain can be thought of as all real number inputs x that give *real* number outputs $f(x)$. It's clear that for a polynomial p, *any* real number input a will generate a real number output $p(a)$. Hence the domain of any polynomial function is all real numbers $\mathbb{R}$. However, domains will become restricted when we perform the following two operations:

1. Division by a variable term: In this case, the denominator may be zero at certain values of x, and these values must be excluded from the domain.

2. An expression involving square roots (or any even root): In this case, the square root of a negative number is imaginary, so we must restrict the domain so that what is under the square root is always non-negative.

3.1.1 Exercise

Find the domain of the following functions *analytically*:

1. $$f(x) = x^3 - 7x^2 + 5x - \pi$$

2. $$f(x) = \frac{x-2}{x^2+2x-3}$$

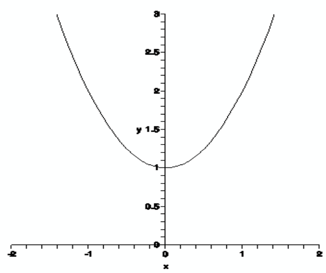
3. $$f(x) = \sqrt{x-1}$$

■

The *range* of a function f is the set of all outputs $f(x)$. This is not always easy to find analytically, but can usually be found graphically by looking at all of the y-values produced by the graph.

3.1.2 Example

The graph of the function $f(x) = x^2 + 1$ is shown below. Clearly, no value less than $y = 1$ shows up on the graph. And all values greater than $y = 1$ show up on the graph (also $f(0) = 1$ is on the graph). Thus the range of f is the interval $[1, \infty)$. To see this analytically, note that x^2 is always ≥ 0, so $x^2 + 1$ is always ≥ 1.



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3.1.3 Exercise

Determine the domain analytically and the range graphically:

1. $$f(x) = \sqrt{x}$$

2. $$f(x) = \sqrt[3]{x}$$

3. The *piecewise defined* function

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 0 \\ -x^2 + 2 & \text{if } x > 0 \end{cases}$$

■

3.2 Function Composition

One way to combine two or more functions into a single function is *composition*. Composition is merely the sequential application of two or more functions. For example, if f and g are given, then the composition function, denoted $f \circ g$, is the function that first applies g to the variable x, and then applies f to the output of g. In other words,

$$(f \circ g)(x) = f\Big(g(x)\Big)$$

We could also apply the functions in the opposite order, first f and then g, in which case we write the composition as $g \circ f$.

3.2.1 Example

Let $f(x) = x^2$ and $g(x) = x - 1$. Then applying g first we have the composition function

$$(f \circ g)(x) = f\Big(g(x)\Big) = f(x-1) = (x-1)^2$$

On the other hand, applying f first gives the composition

$$(g \circ f)(x) = g\Big(f(x)\Big) = g(x^2) = x^2 - 1$$

■

It is important to pay attention to domain considerations when composing functions. Considering the domain of $f \circ g$, we look for all x so that both $g(x)$ is real *and* $f(g(x))$ is real.

3.2.2 Exercise

Compute both compositions $f \circ g$ and $g \circ f$ and also determine their domains.

$$f(x) = \frac{1}{x-1} \qquad\qquad g(x) = x^2 + 1$$

■

3.3 Homework

1. For each function below, find the domain analytically, and find the range graphically:

   (a)
   $$f(x) = \frac{1}{x}$$

   (b)
   $$f(x) = \frac{2}{\sqrt{x^2 - 1}}$$

   (c)
   $$f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0 \\ x^3 + 1 & \text{if } x \geq 0 \end{cases}$$

2. For each pair of functions f and g, form the composition functions $f \circ g$ and $g \circ f$ and determine their domains.

   (a)
   $$f(x) = \sqrt{x} \qquad g(x) = x^2$$

   (b)
   $$f(x) = |x| \qquad g(x) = \sqrt[3]{x - 1}$$

4 Trigonometric Functions

4.1 Basics

An *angle* is formed by two rays (or line segments) intersecting at a single point. The angle begins at the initial side and ends at the terminal side. We measure angles (in standard position) with the initial side along the positive x-axis. Angles in the counter-clockwise direction have positive measure, and those in the clockwise direction have negative measure. One *degree* is $\frac{1}{360}$ of an entire circle, and 2π *radians* is the measure of an entire circle. The corresponding conversion factor is thus $360^\circ = 2\pi$ radians. Unless denoted with the $^\circ$ symbol, all angles will be given in radian measure. Let θ be a given angle, and let (x, y) be any point on the terminal side of θ. Let $r = \sqrt{x^2 + y^2}$ be the distance between the point (x, y) and the origin. We then define the six trig functions of the angle θ as

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

$$\sec\theta = \frac{r}{x} \qquad \csc\theta = \frac{r}{y}$$

Note that the last four of these quantities may be undefined, if x or y in the denominator is zero.

4.1.1 Exercise

Draw an angle of measure $\theta = \dfrac{3\pi}{4}$ and one of measure $\gamma = -540^\circ$. Find a point on the terminal side of each, and calculate the values of all six trig functions. ■

There are a handful of special angles for which you will need to know the values of the six trig functions. We'll start in the first quadrant.

4.1.2 Exercise

Fill in (and *remember*) the following table of special angles

θ	θ (°)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0	0						
$\pi/6$	30						
$\pi/4$	45						
$\pi/3$	60						
$\pi/2$	90						

■

For an angle θ, its *reference angle* θ_R is defined as the smallest angle between the terminal side of θ and the x-axis (and thus $0 \leq \theta_R \leq 90°$). For example, the angle $\theta = 225°$ makes an angle of $\theta_R = 45°$ with the negative x-axis in the third quadrant. The utility of reference angles is that the trig functions of θ_R have the same value (up to plus or minus) as they do at θ. It is useful to note the sign of each of the six trig functions by quadrant. This information, along with the reference angle, make calculating exact values of trig functions easy for many special angles (indeed, any angle whose reference angle is one of those in the exercise above).

4.1.3 Exercise

Using the definitions, list, by quadrant, the sign of each of the six trig functions. ■

4.1.4 Example

To calculate the exact values of $\sin\frac{4\pi}{3}$ and $\cos\frac{4\pi}{3}$, we note that the reference angle for $\theta = \frac{4\pi}{3}$ is $\theta_R = \frac{\pi}{3}$ and that θ is in quadrant 3 (Q3). Both sin and cos are negative in Q3, so

$$\sin\theta = -\sin\theta_R = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\cos\theta = -\cos\theta_R = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

■

4.1.5 Exercise

Find the reference angle and the exact values of $\sin\theta$ and $\cos\theta$ for each of the following angles θ:

$$a)\ \frac{2\pi}{3} \qquad b)\ \frac{7\pi}{4} \qquad c)\ -150^\circ$$

■

4.2 Trig Identities

There are many important and useful trig identities. Below are a few. Note that tan, cot, sec, and csc can be written in terms of sin and cos, so it suffices to calculate the last two of these to find the others.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

4.2.1 Example

Suppose that $\sin\theta = \frac{3}{5}$ and that θ is in quadrant 2. We can find the value of the other five trig functions of θ as follows:

First, we note that cos is negative in quadrant 2 and solve the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ for $\cos\theta$ to get

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Now that we have exact values for $\sin\theta$ and $\cos\theta$, we compute the other four as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{4}{5}} = -\frac{5}{4} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

■

4.3 Solving Trig Equations

We can solve *analytically* any trig equation involving the special angles given in the table above.

4.3.1 Example

To solve the equation $\sin x = -\frac{\sqrt{2}}{2}$, we note that the special angle $\theta = \frac{\pi}{4}$ has $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. We thus want to find all angles x in Q3 or Q4 (since $\sin x < 0$) whose reference angle is $\frac{\pi}{4}$. So the 2 unique solutions x with $x \in [0, 2\pi]$ are $x = \frac{5\pi}{4}$ (Q3) and $x = \frac{7\pi}{4}$ (Q4). To obtain *all* solutions to the given equation, we add (and subtract) all multiples of 2π to the above two solutions to get

$$\frac{5\pi}{4} + 2\pi n \text{ for } n \in \mathbf{Z}$$

$$\frac{7\pi}{4} + 2\pi n \text{ for } n \in \mathbf{Z}$$

■

4.3.2 Exercise

1. Find all solutions (exact values) x to $\cos x = -\frac{1}{2}$ with $x \in [0, 2\pi]$.

2. Find all solutions (exact values) θ to $\tan\theta = \sqrt{3}$ ■

We can also use the *inverse trig functions* (which we study in Calculus 2) to get calculator approximations to solutions of trig equations.

4.3.3 Example

To solve $\cos x = .81$ for $x \in [0, 2\pi]$, we use $x = \cos^{-1}(.81) \approx .627$ radians. Noting that all solutions lie in Q1 or Q4, we have the other solution as $2\pi - .627 \approx 5.66$. ■

4.4 Trig Graphs

If we start with any angle θ and add (or subtract) any multiple of 2π radians, then the new angle has the same value at all six trig functions. We call this behavior *periodicity*. For example, we have that $\sin(\theta+8\pi) = \sin(\theta+2\pi) = \sin\theta$ for any angle θ. The graphs of all six trig functions thus repeat themselves over intervals of length 2π (in fact some repeat over smaller intervals as well). The smallest interval on which a trig function repeats itself is called the *period* of the function.

4.4.1 Exercise

1. Consider the function $f(x) = \sin x$. The domain is $\mathbb{R}$, and the roots are $\pm 0, \pi, 2\pi, 3\pi, \ldots$. The range of f is $[-1, 1]$. The graph has period 2π. Use your calculator to graph two complete cycles of $\sin x$ on the interval $[-2\pi, 2\pi]$.

2. The function $f(x) = \tan x$ has vertical asymptotes (i.e. points *not* in the domain) where $\cos x = 0$, that is $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. The roots occur where $\sin x = 0$. The period is π, and the range of f is $\mathbb{R}$. Use

your calculator to graph two complete cycles of $\tan x$ on the interval $[-\frac{3\pi}{2}, \frac{3\pi}{2}]$. ■

4.5 Homework

1. Find the reference angle and the exact value of all 6 trig functions for the angle $\theta = \dfrac{11\pi}{6}$

2. Given that $\sin\theta < 0$ and $\cos\theta = .4$, find the exact value of the other four trig functions of θ. In what quadrant is θ?

3. Use the identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ to give an exact value for $\sin(15°) = \sin(45° - 30°)$

4. Find the *exact* solutions $x \in [0, 2\pi]$ to the trig equations

   (a) $\tan x = -1$

   (b) $\cos 2x = \frac{\sqrt{3}}{2}$

5. Find *all* values of θ with $\tan\theta = 50$.

6. Using a graphing calculator, graph one complete cycle of the trig functions $\cos x$, $\cot x$, and $\csc x$. Determine the domain, range, vertical asymptotes, and period of each function.

5 Limits at a Point

For a function $f(x)$ and a point $x = a$, we use the following notation

$$\lim_{x \to a^+} f(x) = L$$

to mean that as x gets really really close to (but not necessarily equal to) a from the right (as indicated by the + sign), the function values $f(x)$ get really really close to L. We say that the *right-sided limit* of f at a is L. Similarly, we define the *left-sided limit* of f at a:

$$\lim_{x \to a^-} f(x) = L$$

The two-sided *limit*, (or just plain limit) of f at a is

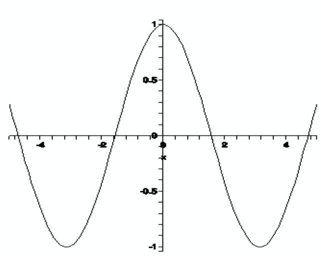
$$\lim_{x \to a} f(x) = L$$

and this limit exists iff both one-sided limits exist and equal L.

5.0.1 Example

1. Consider the trig function $f(x) = \cos x$ at the point $x = 0$. As you trace the curve from the left of $x = 0$, towards $x = 0$, you can see the function values increase towards the value of $f(0) = \cos 0 = 1$. Similarly, from the right as you approach $x = 0$ along the curve, the function values get closer and closer to 1 (at the crest of the curve). Thus we have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = f(0) = 1$$



2. Note that for the piecewise function

$$f(x) = \begin{cases} x+1, & x < 2 \\ -x+3, & x \geq 2 \end{cases}$$

$$\lim_{x \to 2^+} f(x) = f(2) = -2 + 3 = 1$$

$$\lim_{x \to 2^-} f(x) = 3$$

However, since the one-sided limits at $x = 2$ are not equal, the two sided limit $\lim_{x \to 2} f(x)$ does not exist.

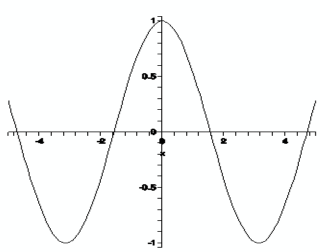
■

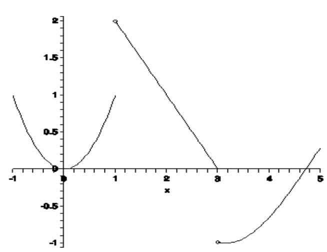
5.1 Estimating Limits Graphically and Numerically

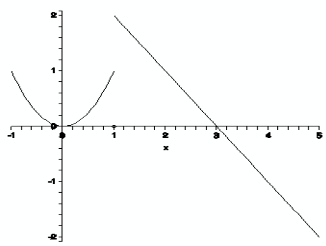
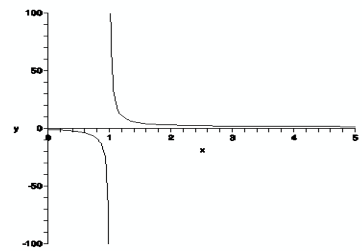
As shown in the example $f(x) = \cos x$ above, if a graph is *smooth* at a point (whatever that means!), then the limit is most likely also the function value. On the other hand, the piecewise function above shows that graphs with gaps (or breaks or holes) do not behave so properly. When trying to estimate a limit graphically, trace the graph from either side of the point of interest $x = a$. As you get closer to the point a, what are the function values doing? Do they approach some fixed value? If so, this is your one-sided limit.

5.1.1 Exercise

For each graph below, determine the one-sided limit of the function at $x = 1$ and $x = 3$. Also give the function values at these points, and the two-sided limits.





5.1.2 Exercise

Use the Table and Graphing capabilities of your calculator to estimate the following limits:

1. $\lim_{x \to 0} \frac{\sin x}{x}$
2. $\lim_{x \to 0} \sin \frac{1}{x}$ ■

5.2 Computing Limits Analytically

How can we determine if a function has a hole or break in its graph analytically? Normally, breaks are introduced by using a piecewise function. Holes often occur when division by zero occurs. So absent these two characteristics, most functions have the desirable property of smoothness. Indeed, most of the functions we will study often have the property that the two-sided limit at a point is equal to the function value at that point. The key exceptions are those two above (piecewise functions and division by zero) and vertical asymptotes where functions aren't defined. We collect this information in the following

5.2.1 Classification of Limits

1. If $p(x)$ is a polynomial, then for any point $x = a$

$$\lim_{x \to a} p(x) = p(a)$$

2. If $f(x) = \frac{p(x)}{q(x)}$ is a rational function, and $q(a) \neq 0$, then

$$\lim_{x \to a} f(x) = f(a)$$

3. If $f(x)$ is a trig function and $x = a$ is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

■

5.2.2 Exercise

Find the following limits analytically:

1.
$$\lim_{x \to 5.1} 2x^2 - 7x + 3$$

2.
$$\lim_{x \to 0} \frac{2x^2 - 7x + 3}{x^3 - 7}$$

3.
$$\lim_{x \to 1} \sin x$$

So where do problems arise when trying to compute limits analytically? In examining the cases above, we see that if we can plug $x = a$ into the function f and compute $f(a)$, then it's likely that this is the limit. When can't we compute $f(a)$? When a is not in the domain of f. Examining the cases above more closely leads to the following:

1. If $f(x) = \frac{p(x)}{q(x)}$ is a rational function, and $q(a) = 0$, then a is not in the domain of f. There are two further possibilities:

   (a) If $p(a) = 0$ (i.e. if $f(a)$ is of the form $\frac{0}{0}$), then we can factor $(x - a)$ out of both the numerator and denominator and cancel it. We then continue to try and evaluate $f(a)$ until all common factors are cancelled.

(b) If $p(a) \neq 0$ (i.e. if $f(a)$ is of the form $\frac{k}{0}$ with $k \neq 0$), then f has a vertical asymptote at $x = a$. Further discussion of this type of limit is in the next section.

2. If $f(x)$ is a trig function and a is not in the domain of f, then again, f has a vertical asymptote at $x = a$.

5.2.3 Example

To compute the limit

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Set $f(x) = \frac{x^2-1}{x-1}$. If we evaluate the function at $a = 1$, we get the value $f(1) = \frac{0}{0}$. This is explained in case 1a) above. So we should be able to factor $(x - 1)$ out of both the numerator and denominator, cancel it, and try again.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} \frac{x + 1}{1}$$

When evaluating this last expression at $a = 1$, we get $\frac{2}{1} = 2$ so we have

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

If you look at the graph of f, it appears to be a smooth line. However, there is a hole in the graph at $x = 1$ and $f(1)$ does not exist. ■

5.3 Infinite Limits and Vertical Asymptotes

If, as x approaches a from the right, the function values $f(x)$ grow without bound (in the positive or negative direction), then we say that the limit of f at a is infinite, and write

$$\lim_{x \to a^+} f(x) = \infty$$

Indeed, this behavior is displayed by functions with a vertical asymptote at $x = a$. However, we must distinguish both between approaching a from the right and left, and whether the corresponding limit is positive or negative infinity.

5.3.1 Vertical Asymptotes of Rational Functions

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function and assume that p and q have no common roots. Notice that if a is a root of q, then $q(a) = 0$ but $p(a) \neq 0$. Thus $f(a) = \dfrac{p(a)}{0}$ is undefined. In this case we say that f has a *vertical asymptote at* $x = a$.

5.3.2 Exercise

Let $f(x) = \dfrac{x+1}{x-1}$. Then f has a vertical asymptote at $x = 1$. Use the table capability of your graphing calculator to investigate what happens to $f(x)$ for x very close to 1. Does this behavior change depending on which side of 1 you are on? What are $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$?

Now graph $f(x)$ on your graphing calculator and verify the two limits above and the vertical asymptote at $x = 1$ ■

If f has a vertical asymptote at $x = a$, then we see that the values of $f(x)$ get arbitrarily large (in the positive or negative direction) as x gets closer to a. As in the exercise above, we see that this behavior may change from $+\infty$ to $-\infty$, depending on which direction we approach a from.

Using our limit notation, we then have one of the following situations:

$$\lim_{x\to a^+} f(x) = \pm\infty$$

$$\lim_{x\to a^-} f(x) = \pm\infty$$

We can determine these *one-sided* limits *analytically* merely by determining the *sign* of f near a (we can also use the table function of the calculator to verify this).

5.3.3 Exercise

Determine, *analytically*, both $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ for the following functions and verify these limits graphically:

1. $f(x) = \dfrac{x+1}{x-1}$

2. $f(x) = \dfrac{x+1}{(x-1)^2}$ ■

Graphically, a vertical asymptote at a is represented by the vertical line $x = a$. The graph of f then gets closer and closer to this line and approaches $\pm\infty$ as x gets closer to a.

5.3.4 Exercise

Use your graphing calculator to find all vertical asymptotes for the function

$$f(x) = \frac{x+7}{x^2 - x - 6}$$

For each vertical asymptote, determine the corresponding one-sided limits *graphically*. ■

5.3.5 Vertical Asymptotes of Trig Functions

We have seen that the four trig functions $\tan x$, $\cot x$, $\sec x$ and $\csc x$ all have vertical asymptotes periodically.

5.3.6 Exercise

Find the following limits:

1. $$\lim_{x \to \frac{\pi}{2}^+} \tan x$$

2. $$\lim_{x \to \pi^+} \csc x$$

3. $$\lim_{x \to \frac{\pi}{2}^-} \cos x$$

6 Limits at Infinity

6.1 End Behavior of Polynomials

We talk about the *end behavior* of polynomials. In other words, what happens to the graph of a polynomial $p(x)$ as x gets *really* big (in both the positive and negative directions). We can start by looking at *monomials* of the form $p(x) = x^n$. Here, the end behavior is completely determined by whether n is even or odd. For if n is even, then x^n is positive for both positive and negative values of x. On the other hand, if n is odd, x^n is negative if x is negative, and x^n is positive if x is positive.

6.1.1 Exercise

1. On your graphing calculator, graph the polynomials x^2, x^4, and x^6 in the same viewing window and note the end behavior.

2. Do the same for x^1, x^3, and x^5. ■

So for n even, the graph of $p(x) = x^n$ goes up as x gets big in both the positive and negative directions.

If n is odd, then the graph of $p(x) = x^n$ goes up as x gets big in the positive direction, but goes down as x gets big in the negative direction.

We again use the term *limit* to describe this end behavior.

6.1.2 Example

1. Since x^6 goes *up* to positive infinity as x gets big in the *negative* direction (i.e. as x goes to minus infinity), we write

$$\lim_{x\to-\infty} x^6 = \infty$$

and say that the limit of x^6 as x goes to negative infinity is positive infinity. Similarly, we have that

$$\lim_{x\to\infty} x^6 = \infty$$

2. For the odd degree monomials, we have

$$\lim_{x\to-\infty} x^5 = -\infty \quad \text{and} \quad \lim_{x\to\infty} x^5 = \infty$$

■

If we put a positive coefficient (other than 1) in front of our monomial, all it does to the graph is *stretch* or *shrink* it, so the end behavior is unchanged.

6.1.3 Exercise

1. Graph x^2, $5x^2$, and $\frac{1}{2}x^2$ in the same viewing window. What are $\lim_{x\to-\infty} 5x^2$ and $\lim_{x\to\infty} 5x^2$?
2. Graph x^3, $5x^3$, and $\frac{1}{2}x^3$ in the same viewing window. What are $\lim_{x\to-\infty} 5x^3$ and $\lim_{x\to\infty} 5x^3$? ■

If we put a negative coefficient in front of our monomial p, this will similarly shrink or stretch the graph of p, but also will rotate the graph of p about the x-axis.

6.1.4 Exercise

1. Graph x^2, $-5x^2$, and $-\dfrac{1}{2}x^2$ in the same viewing window. What are $\lim_{x\to-\infty} -5x^2$ and $\lim_{x\to\infty} -5x^2$?
2. Graph x^5, $-2x^5$, and $-\dfrac{1}{3}x^5$ in the same viewing window. What are $\lim_{x\to-\infty} -2x^5$ and $\lim_{x\to\infty} -2x^5$? ■

What about the end behavior of those polynomials with more than one term? Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be an n^{th} degree polynomial. In a general sense, the end behavior of p is completely determined by the leading term $a_n x^n$. In other words, as x gets really big (in the positive or negative directions), the term $a_n x^n$ gets much "bigger" than all of the other terms combined and dominates the end behavior of p. So we have

$$\lim_{x \to -\infty} p(x) = \lim_{x \to -\infty} a_n x^n$$

and

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n$$

and we can use the above results on monomials to determine the appropriate limits.

6.1.5 Exercise

Find the following limits without a graphing calculator:

1. $\lim_{x \to -\infty} 3x^7 + 7x^5 - 2x^4 + 3x^2 - 1$
2. $\lim_{x \to \infty} -2x^6 + 7x^3 - x^2 + 1$ ■

6.2 End Behavior of Rational Functions and Horizontal Asymptotes

What about the end behavior of rational functions? We can treat the numerator and denominator separately, and use our facts from polynomial end behavior. So let

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

be a rational function with p and q polynomials of degree n and m respectively. We know that the end behaviors of p and q are completely determined by the leading terms, so the end behavior of f is likewise determined by the ratio of the leading terms. That is,

$$\lim_{x\to\pm\infty} f(x) = \frac{\lim\limits_{x\to\pm\infty} p(x)}{\lim\limits_{x\to\pm\infty} q(x)} = \frac{\lim\limits_{x\to\pm\infty} a_n x^n}{\lim\limits_{x\to\pm\infty} b_m x^m} = \lim_{x\to\pm\infty} \frac{a_n}{b_m} x^{n-m}$$

So by comparing the degrees of p and q there are then just three cases to consider: $n > m$, $n = m$, $n < m$ as shown in the following exercise.

6.2.1 Exercise

Determine the following limits *analytically*. When you are done, summarize the three different cases.

1. $$\lim_{x\to-\infty} \frac{3x^3 - 2x^2 + 7x - 1}{-2x^2 + 5x - 8}$$

2. $$\lim_{x\to\infty} \frac{-2x^4 + 9x^3 + 16x - 10}{7x^4 + 3x^3 + 8}$$

3. $$\lim_{x\to\infty} \frac{2x^3 + 9x^2 - x + 1}{8x^4 + 5x^3 - 7x^2 + x - 8}$$

■

In case one above, the rational function has an infinite limit and thus the end behavior of the graph goes up or down as do polynomials. However, in the other cases, f has a finite limit, and the graph will approach this limiting value as a *horizontal asymptote*.

6.2.2 Exercise

For each of the three functions in the exercise above, use your graphing calculator to observe any horizontal asymptotes. Write the equation of these horizontal lines. ■

6.2.3 Exercise

Considering that the 6 trig functions are periodic, what might be their limits at infinity? Use your graphing calculator to investigate. ∎

7 Continuity

The concept of *continuity of a function at a point* involves a closer look at one-sided limits and the corresponding graphical behavior.

7.0.4 Example

1. For the piecewise function

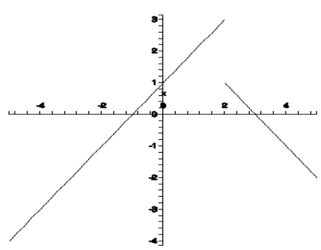
$$f(x) = \begin{cases} x+1, & x < 2 \\ -x+3, & x \geq 2 \end{cases}$$

   we have seen that

$$\lim_{x \to 2^+} f(x) = 1$$

$$\lim_{x \to 2^-} f(x) = 3$$

   Since the one-sided limits do not agree, the two-sided limit does not exist. Graphically, this behavior is shown below by the *break* in the graph of f at $x = 2$.



2. For the rational function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

   the left and right-sided limits exist and are equal, so that the two-sided limit also exists, i.e.

$$\lim_{x \to 2} f(x) = 4$$

   BUT NOTE that $f(2)$ does not exist. This is described graphically as a hole in the graph of f at the point $x = 2$.

■

In a graphical sense, for a function to be continuous at a point, one should be able to draw the graph of the function through that point without lifting pencil from paper. The functions above are not continuous at the point $x = 2$. So the requirements for continuity are that the two-sided limit exists (i.e. no breaks in the graph) and the two-sided limit equals the function value (i.e. no holes in the graph). This can be stated analytically by the following:

7.0.5 Definition

A function $f(x)$ is *continuous* at a point $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

■

If f is not continuous at $x = a$, it is said to be *discontinuous* there.

7.0.6 Example

In the previous example, we see the two standard types of discontinuities, *essential* and *removable*.

1. The piecewise function

   $$f(x) = \begin{cases} x + 1, & x < 2 \\ -x + 3, & x \geq 2 \end{cases}$$

   has an essential discontinuity at $x = 2$ since the one-sided limits do not agree. The term essential is used because the discontinuity is difficult to correct.

2. The function

   $$f(x) = \frac{x^2 - 4}{x - 2}$$

   however, has a removable discontinuity at $x = 2$. This terminology implies that we can correct the discontinuity rather easily by merely redefining the function differently at a single point. Indeed, if we leave the rest of f alone and define $f(2) = 4$, then f would be continuous at $x = 2$. This function could be represented as the piecewise function

$$g(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

■

A function with a vertical asymptote at $x = a$ also has a discontinuity there, for we have seen that in this case the one-sided limits are infinite, and moreover, the function fails to exist at a.

7.1 Continuity on Intervals

Continuity is a desirable property in that a continuous function has a *smooth* graph. We talk about continuity on an open interval (a, b) to mean that the function is continuous at every point between a and b (excluding a and b themselves).

7.1.1 Exercise

Find open intervals on which the function $f(x) = \tan x$ is continuous. ■

A function is said to be *continuous everywhere* if it is continuous at every point on the entire real line $(-\infty, \infty)$.

We have seen that for a polynomial $p(x)$, at any point a, we have

$$\lim_{x \to a} p(x) = p(a)$$

so that **polynomials are continuous everywhere**. Similarly for rational functions and trig functions, as long as there is no vertical asymptote, the functions are continuous. In other words, **rational functions and trig functions are continuous wherever they are defined**. In particular, $\sin x$ and $\cos x$ are continuous everywhere.

We say that a function f is continuous on the *closed interval* $[a, b]$ provided it is continuous on (a, b) and furthermore

$$\lim_{x \to a^+} f(x) = f(a)$$

and

$$\lim_{x \to b^-} f(x) = f(b)$$

These last two conditions are called *left-continuity* and *right-continuity* respectively. Continuity on closed intervals is important in the following useful theorem:

7.2 Intermediate Value Theorem (IVT)

Suppose f is continuous on the closed interval $[a, b]$. Let k be any number between $f(a)$ and $f(b)$. Then there exists a number c in the interval $[a, b]$ with $f(c) = k$. ■

In other words, if f is continuous on the closed interval $[a, b]$, then every value k *intermediate* to the endpoint function values $f(a)$ and $f(b)$ is taken on (i.e. there is some c with $f(c) = k$).

A graphical interpretation of the IVT is that to get *smoothly* from $f(a)$ to $f(b)$ on the graph of f, one must cross *every* horizontal line $y = k$ along the way.

7.2.1 Example

Consider the function $f(x) = x^2$ on the closed interval $[0, 2]$. Since f is continuous everywhere, the IVT applies. Thus, f takes on every value between $f(0) = 0$ and $f(2) = 4$ on the interval $[0, 2]$. In particular, there is some point $c \in [0, 2]$ with $f(c) = 3$. To find the point c, we solve $f(x) = x^2 = 3$ to see that $c = \sqrt{3} \in [0, 2]$ ■

7.2.2 Exercise

Sketch graphs of examples of functions that do not satisfy the IVT. Try finding an example with an essential discontinuity and one with a removable discontinuity. In each case, identify the value(s) k that doesn't get taken on by f.

8 Tangent Lines and Derivatives

What is a tangent line? Some are familiar with a tangent to a circle, which is a line that touches the circle at exactly one point. For a general curve given by a function $f(x)$, we can think of the tangent line to f at the point $x = a$ as the line that goes through the point on the curve $(a, f(a))$ and whose slope "best represents the slope of the curve itself". But what do we mean by "best represents"?

To be more explicit, we need to give a definition of this tangent line. But first, let's see how we might approximate it with secant lines.

8.1 Secant Lines

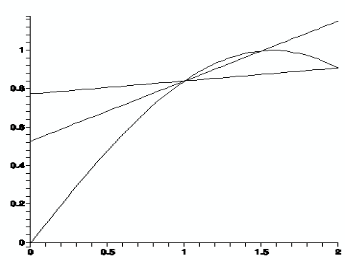
Let $f(x)$ be a function and fix the point $x = a$. So the coordinates of the corresponding point on the graph of f are given by $(a, f(a))$. If we take another point $(x, f(x))$ on the graph of f, then the line through these two points is called a *secant line* and has slope

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

In general, a secant line is any line that goes through two points on the curve. The first important idea here is that a secant line serves as an approximation to the tangent line at a, and would be a better approximation if x were closer to a. This is illustrated in the following:

8.1.1 Example

We are interested in the tangent line to the graph of $f(x) = \sin x$ at the point $a = 1$. Two different secant line approximations to this tangent line are shown below.



The first secant line goes through the points $(2, f(2))$ and $(1, f(1))$ so has slope

$$m_{sec} = \frac{f(2) - f(1)}{2 - 1} = \frac{\sin 2 - \sin 1}{1} \approx .068$$

For the second secant line we move the right point closer to $a = 1$ to get a better approximation to the tangent line at $a = 1$. This second secant line goes through the points $(1.5, f(1.5))$ and $(1, f(1))$ so has slope

$$m_{sec} = \frac{f(1.5) - f(1)}{1.5 - 1} = \frac{\sin 1.5 - \sin 1}{.5} \approx .312$$

If we were to continue moving the right endpoint of our secant lines closer to $a = 1$ (say $x = 1.4, 1.3, \ldots 1.1, 1.01, 1.001, \ldots$), then our approximations would get better and better. ■

8.1.2 Exercise

We wish to approximate the tangent line to the graph of $f(x) = x^2$ at the point $a = 0$ using secant lines. Compute the slopes of the secant lines through the following points:

1. $(.5, f(.5))$ and $(0, f(0))$
2. $(.1, f(.1))$ and $(0, f(0))$
3. $(.01, f(.01))$ and $(0, f(0))$
4. Look at the graph of f. What do you think the slope of the tangent line is?

■

So the closer we take our second endpoint to the point $x = a$, the better our approximation is for the tangent line slope. And if we take this point *closer and closer* to $x = a$, the approximation gets better and better. We have another term for *closer and closer*. We call it the **limit**.

8.2 Definition of Tangent Line

The *slope of the tangent line* to the graph of $f(x)$ at the point $x = a$ is given by

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Since this tangent line goes through the point $(a, f(a))$ on the graph of f, we use the point-slope form to write the *equation of the tangent line*

$$y - f(a) = m_{tan}(x - a)$$

8.2.1 Exercise

Use the above definition to compute the slope of the tangent line to $f(x) = x^2$ at $a = 0$. Then write the equation of this tangent line. ■

8.3 Other Notations

Above, we use the notation m_{tan} for the slope of the tangent line to f at the point $x = a$. This slope is also called the **derivative** of f at $x = a$ and is denoted by $f'(a)$. Thus we can state the following:

8.3.1 Definition of Derivative

The derivative of the function f at the point $x = a$ is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Graphically, $f'(a)$ represents the slope of the tangent line to f at $x = a$. ■

Computing derivatives by calculating limits is often difficult. Indeed, an initial glance at what is called the *difference quotient*

$$\frac{f(x) - f(a)}{x - a}$$

shows that letting x approach a gives a value of zero in the denominator. This typically gives rise to computing limits of the form $\frac{0}{0}$. By making a subtle substitution we can construct an alternative version of the difference quotient that is often easier to work with.

8.3.2 Alternative Definition of Derivative

In the definition above, we replace x with $a+h$. If we set $x = a+h$, then letting $x \to a$ is equivalent to letting $h \to 0$. The difference quotient becomes

$$\frac{f(x)-f(a)}{x-a} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

We thus have the alternative definition of the derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

8.3.3 Exercise

Use the alternative definition to compute $f'(1)$ for $f(x) = x^2$ ■

The *derivative function* $f'(x)$ of f is the function that would give the derivative at *any* point $x = a$. It is often more convenient to compute the derivative function and then use it to compute tangent line slopes at specific points $x = a$. To compute the derivative function, we just use the definition applied to an *arbitrary* point x instead of a specific numerical value a.

8.3.4 Definition of Derivative Function

The derivative function of f is the function $f'(x)$ given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(Note that we could also use the original version of the difference quotient to compute $f'(x)$) ■

8.3.5 Example

We compute the derivative function $f'(x)$ for the function $f(x) = x^2$, and use it to find the slope of the tangent line to f at various points.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0}(\frac{2xh}{h} + \frac{h^2}{h}) = \lim_{h\to 0}(2x + h) = 2x$$

So $f'(x) = 2x$ and we use this to see that the slope of the tangent line to f at

1. $x = 0$ is $f'(0) = 2 \cdot 0 = 0$
2. $x = 1$ is $f'(1) = 2 \cdot 1 = 2$
3. $x = -50$ is $f'(-50) = 2 \cdot (-50) = -100$ (wow, that's steep!)

■

8.3.6 Exercise

Compute the derivative function $f'(x)$ for $f(x) = x^2 - 1$. Then find the slope of the tangent line to f at $x = 1$ and write the equation of this line. ■

Finally, we introduce the Leibniz notation for the derivative. If we let $y = f(x)$, then we write

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x))$$

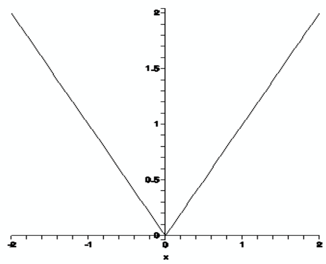
$$f'(a) = \frac{dy}{dx}\bigg|_{x=a}$$

8.4 Differentiability

We have seen that it is possible that a limit may not exist or may be infinite. Thus this could also happen in the case of derivatives, since they are limits. We say that the function f is *differentiable* at a point $x = a$ if $f'(a)$ exists. Notice that differentiability of a function implies that it has a tangent line. So when might a function not be differentiable? In other words, when might a function not have a tangent line? The following classic example describes this situation both analytically and graphically.

8.4.1 Example

Let $f(x) = |x|$ be the absolute value function shown below. What do you suppose the tangent line slope is at $x = 0$?



Notice that this function can be thought of as a piecewise function consisting of two lines, namely

$$f(x) = |x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$$

Fixing one point at $x = 0$, we see the following: Any secant line with the other point to the right of $x = 0$ has slope 1 (it is the line $y = x$) and any secant line with the other point to the left of $x = 0$ has slope -1. Since our definition of tangent line slope requires a two-sided limit of secant line slopes, this is problematic. To see this analytically, we use the piecewise definition to compute the one-sided limits:

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} (1) = 1$$

These values agree with our graphical interpretation above and show that the two-sided limit cannot exist. Thus $f'(0)$ does not exist and f is not differentiable at $x = 0$. ∎

It turns out that if a function if differentiable at a point then it is also continuous there. We state this formally as:

8.4.2 Differentiability Theorem

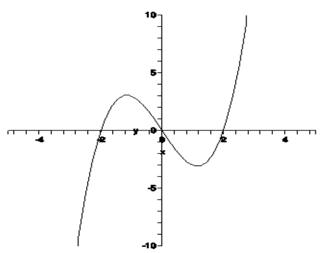
If f is differentiable at $x = a$, then f is continuous at $x = a$. ∎

The contrapositive of this statement (which is also a theorem) is that if a function is not continuous at a point, then it is not differentiable there either. This makes sense, since for example if a function has a break in the graph then it clearly can have no tangent line there.

The converse of this theorem, however, is not true. The absolute value example above shows a function and a point where the function is clearly continuous, but not differentiable.

8.4.3 Exercise

Suppose the graph of f is given below. By tracing the curve with a straight-edge, we can approximate the slope of the tangent line at each point x. We can collect this information in order to sketch a graph of the derivative function $f'(x)$. Do this.



■

9 Velocity

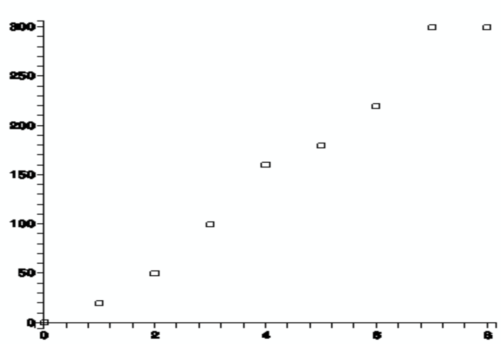
The concept of the derivative is seen graphically as the slope of the tangent line to a curve at a point. The concept of derivative also has a physical interpretation involving motion and velocity. We investigate this interpretation in the following example:

9.1 Motion along a straight line

Suppose an automobile is moving along a straight road. Suppose at time t (measured in hours) that the car is $f(t)$ miles away from it's original starting point. We denote the starting point as $f(0) = 0$. The table below gives a series of measurements for $f(t)$ at hourly intervals.

t	0	1	2	3	4	5	6	7	8
$f(t)$	0	20	50	100	160	180	220	300	300

If we plot t along the x-axis and $f(t)$ along the y-axis, we get the following set of points, called the *position versus time graph*:



9.1.1 Average Velocity

We can compute *average* velocities over time intervals. For example, in the first hour of the trip (i.e. from $t = 0$ to $t = 1$), the car travels 20 miles, so that the average velocity over this time interval is

$$v_{avg} = \frac{f(1) - f(0)}{1 - 0} = \frac{20 - 0}{1 - 0} = 20 \text{ mph}$$

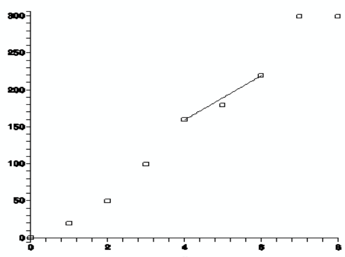
More generally, the average velocity over the time interval $[t_1, t_2]$ is given by

$$v_{avg} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

So the average velocity of the car over the time interval $[4, 6]$ is

$$v_{avg} = \frac{f(6) - f(4)}{6 - 4} = \frac{220 - 160}{2} = 30 \text{ mph}$$

Returning to the position versus time curve, we see that the average velocity of 30 mph is just the slope of the secant line between the two points $(4, f(4))$ and $(6, f(6))$ as shown below.



More generally, we see that the average velocity over the interval $[t_1, t_2]$ is the slope of the secant line between the points $(t_1, f(t_1))$ and $(t_2, f(t_2))$ on the position versus time curve.

9.1.2 Instantaneous Velocity

What is the *instantaneous velocity* of the car at a particular time t_1? This is effectively what the speedometer on the car is trying to measure. If we fix t_1 and compute average velocities over intervals $[t_1, t_2]$ where t_2 gets closer and closer to t_1, then we would get better approximations to the instantaneous velocity at t_1. Of course, we can't really do this for our example because we only have data at hourly intervals. But if we "connected the dots" on our position versus time curve to generate a continuous curve, then we could make graphical approximations to the instantaneous velocity. We will do this in the next example. But for now, we give the formal definition:

9.1.3 Definition

The *instantaneous velocity* at time t_1 is given by

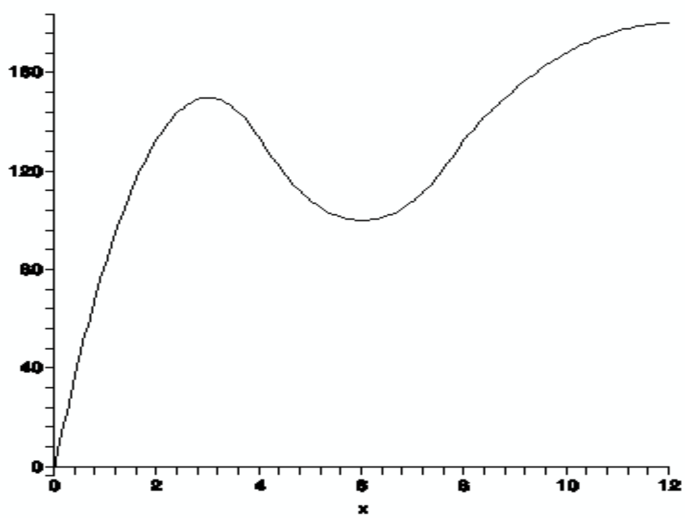
$$v_{inst} = \lim_{t_2 \to t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Using our alternative notation, we have

$$v_{inst} = \lim_{h \to 0} \frac{f(t_1 + h) - f(t_1)}{h}$$

9.1.4 Exercise

A car travels on a straight road with position versus time curve given below:



1. Estimate the average velocity between $t = 0$ and $t = 3$
2. Estimate the average velocity on the interval $[4, 12]$
3. Estimate the instantaneous velocity at $t = 3$
4. Estimate the instantaneous velocity at $t = 9$
5. What is the car doing on the interval $[3, 6]$?

10 Shortcuts for Computing Derivatives

While computing derivatives using the limit definition is fun, it is quite often difficult or tedious to do. As such, we have developed *shortcuts* for computing derivatives. These shortcuts are just rules that allow us to compute derivatives without using the limit definition. However, if we were to formally prove these rules, we would have to revert to the limit definition. We will not do that here; we merely state and apply these shortcuts.

10.1 Power Rule

For n any real number

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

10.1.1 Example

Let $f(x) = x^3$ and $g(x) = x^{98}$. Then $f'(x) = 3x^2$ and $g'(x) = 98x^{97}$ ∎

10.2 Linearity Rules

1. $$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

2. $$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

3. $$\frac{d}{dx}(k \cdot f(x)) = k \cdot f'(x) \text{ for k a constant}$$

∎

10.2.1 Example

$$\frac{d}{dx}(3x^3-7x) = \frac{d}{dx}(3x^3)-\frac{d}{dx}(7x) = 3\frac{d}{dx}(x^3)-7\frac{d}{dx}(x) = 3(3x^2)-7(1x^0) = 9x^2-7$$

■

Note that with the combination of these rules above, we are able to quickly compute the derivative of **any** polynomial function.

10.2.2 Exercise

Compute $f'(x)$ for the following functions:

1. $f(x) = x - 5$
2. $f(x) = 3x^7 - 5x^2 + 7x - 2$
3. $f(x) = 3\sqrt{x} - \frac{1}{x^2}$

10.3 Higher Order Derivatives

We define the higher order derivatives of a function $y = f(x)$ as follows:

$$f''(x) = \frac{d}{dx}(f'(x)) = \text{ the second derivative of f}$$

$$f'''(x) = \frac{d}{dx}(f''(x)) = \text{ the third derivative of f}$$

$$f^{(4)}(x) = \frac{d}{dx}(f'''(x)) = \text{ the fourth derivative of f}$$

and in general

$$f^{(n)}(x) = \frac{d}{dx}(f^{(n-1)}(x)) = \text{ the n}^{\text{th}}\text{ derivative of f}$$

The Leibniz notation in this case is

$$f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n}(f(x))$$

10.3.1 Exercise

Compute the second derivative $f''(x)$ for the function $f(x) = 2x^2 + \sqrt{x} + \dfrac{1}{x}$

■

10.4 Product and Quotient Rules

The linearity rules show that the *derivative of a sum is the sum of the derivatives.* Is this true if we replace *sum* with *product*? The answer is no, as the example below shows.

10.4.1 Example

Let $g(x) = x^2$ and $h(x) = x^3$. Then the product of g and h is the function $F(x) = x^5$. The *derivative of the product* is thus $F'(x) = 5x^4$. On the other hand, the *product of the derivatives* is $g'(x) \cdot h'(x) = 2x \cdot 3x^2 = 6x^3 \neq F'(x)$ ■

10.4.2 Product Rule

$$\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = f(x)g'(x) + g(x)f'(x)$$

■

Similarly, the *derivative of a quotient* is not equal to the *quotient of the derivatives*. Instead, we have the:

10.4.3 Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

■

10.4.4 Example

1.

$$\frac{d}{dx}\left(\frac{1}{x^2}\cdot(x^3-7x)\right) = x^{-2}\cdot\frac{d}{dx}(x^3-7x) + (x^3-7x)\cdot\frac{d}{dx}(x^{-2})$$
$$= x^{-2}\cdot(3x^2-7) + (x^3-7x)\cdot(-2x^{-3})$$

2.

$$\frac{d}{dx}\left(\frac{x^3}{x^2-7}\right) = \frac{(x^2-7)\cdot\frac{d}{dx}(x^3) - (x^3)\cdot\frac{d}{dx}(x^2-7)}{(x^2-7)^2}$$
$$= \frac{(x^2-7)\cdot 3x^2 - x^3\cdot 2x}{(x^2-7)^2}$$

10.4.5 Exercise

Differentiate the following functions:

1.

$$y = (\frac{1}{x} + \sqrt{x})(x^3 - 7x)$$

2.

$$y = \frac{x^3+4x}{2x^4+7x}$$

■

10.5 Trig Derivatives

We first state the derivatives of the sine and cosine functions, then use the quotient rule to obtain the derivatives of the other four trig functions.

10.5.1 Derivatives of sine and cosine

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

■

10.5.2 Exercise

Use the quotient rule to differentiate

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

■

10.5.3 Other Trig Derivatives

1. $$\frac{d}{dx}\tan x = \sec^2 x$$

2. $$\frac{d}{dx}\cot x = -\csc^2 x$$

3. $$\frac{d}{dx}\sec x = \sec x \tan x$$

4. $$\frac{d}{dx}\csc x = -\csc x \cot x$$

10.6 Chain Rule

If $H(x)$ can be written as a composition of two functions $H(x) = (f \circ g)(x)$, then the derivative of H is gotten by the chain rule:

$$\frac{d}{dx}(H(x)) = f'(g(x))g'(x)$$

10.6.1 Example

Let $H(x) = \sin(x^2 + x)$ so that $H(x) = (f \circ g)(x)$ where

$$g(x) = x^2 + x$$

$$f(x) = \sin x$$

Computing the derivatives of f and g we have

$$f'(x) = \cos x$$

$$f'(g(x)) = \cos(x^2 + x)$$
$$g'(x) = 2x + 1$$

so that

$$H'(x) = f'(g(x))g'(x) = \cos(x^2 + x) \cdot (2x + 1) = (2x + 1)\cos(x^2 + x)$$

■

10.6.2 Exercise

Identify each function $H(x)$ as a composition and use the chain rule to compute $H'(x)$

1. $$H(x) = (3x^2 + 2x - 1)^6$$

2. $$H(x) = \sin(\frac{1}{x})$$

3. $$H(x) = \sin^2(\cos x)$$

   Hint: You might want to write H as a composition of three functions and use the chain rule twice

11 Implicit Differentiation

11.1 Example

The equation of a circle of radius one centered at the origin is given by $x^2 + y^2 = 1$. Notice that this equation does **not** describe a function as it cannot be solved *explicitly* for y in terms of x. However, the circle still certainly has tangent lines and we would like to compute them. How can we do this? There are two approaches:

1. Cut the circle in half, and treat each half separately as a function of x. The top half can be described by $f(x) = \sqrt{1-x^2}$ and the bottom half can be described by $g(x) = -\sqrt{1-x^2}$. If we are interested in tangent lines on the top half, for example, we could *explicitly* compute $f'(x)$ using the chain rule:

$$f'(x) = \frac{d}{dx}(1-x^2)^{\frac{1}{2}} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

2. The second method is to differentiate both sides of the equation $x^2 + y^2 = 1$ with respect to x, treating y *implicitly* as a function of x. This means that anytime we differentiate an expression involving y, we must apply the chain rule. Here is how it works:

$$\frac{d}{dx}(x^2 + y^2 = 1)$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y\frac{d}{dx}y = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

If we solve this last equation for $\frac{dy}{dx}$ we get that

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Notice that by letting $y = f(x)$ we see that the two methods agree.

Finally, we can compute tangent line slopes at various points on the circle:

(a)

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{-0}{1} = 0$$

(b)

$$\left.\frac{dy}{dx}\right|_{(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$$

(c)

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{-1}{0} = \text{ vertical tangent}$$

11.1.1 Technique of Implicit Differentiation

In general, *any* equation in two variables x and y describes a curve in the plane (i.e. all points (x, y) satisfying the equation). To find tangent line slopes on such a curve;

1. Differentiate both sides of the equation with respect to x.
2. Treat y *implicitly* as a function of x, thus using the chain rule whenever differentiating any expression involving y.
3. Solve the new equation for $\frac{dy}{dx}$ to give an expression for the tangent line slope for points on the curve (x, y).

11.1.2 Exercise

1. Find $\frac{dy}{dx}$ for the parabola $y^2 = x$. Use this to find the slope of the tangent line to the parabola at the points $(9, 3)$ and $(9, -3)$.
2. Find $\frac{dy}{dx}$ for $\sqrt{y} - y \sin x = 2$.

12 Monotonicity

A function is *monotonic* on an interval I if it is either increasing, decreasing, or constant on I.

Increasing (decreasing) on I means that as you move from left to right along the x-axis, the function values get bigger (smaller). In this case, the graph should move up (down) as we move left to right. The function is constant on I if the function values are all equal.

Since the derivative at a point is measuring the *slope* of the tangent line, and we know that positive (negative) sloped lines move up (down) as we move left to right, the following theorem should be no suprise.

12.0.3 Monotonicity Theorem

1. $$\text{If } f'(x) > 0 \text{ for all } x \in (a,b) \text{ then } f \text{ is increasing on } (a,b)$$
2. $$\text{If } f'(x) < 0 \text{ for all } x \in (a,b) \text{ then } f \text{ is decreasing on } (a,b)$$
3. $$\text{If } f'(x) = 0 \text{ for all } x \in (a,b) \text{ then } f \text{ is constant on } (a,b)$$

■

To apply this theorem, we wish to find all intervals on which f is monotonic. To do this, we need only find all intervals on which f' has a constant sign. We first find the roots of f', and subdivide the entire real line into intervals whose endpoints are successive roots. If f' is continuous, then it will only change sign at a root. Thus between any two consecutive roots a and b of f', f' must have a constant sign. We pick a test point c in (a,b) and compute the sign of $f'(c)$. This is then the sign of f' on the entire interval. The entire sign chart of f' then tells us the monotonic behavior of f. This is illustrated in the following:

12.0.4 Example

For $f(x) = x^3 - 3x^2 + 1$, we first find the roots of f' by solving $f'(x) = 3x^2 - 6x = 0$. The real roots are $x = 0$ and $x = 2$. Thus we subdivide the real line into the three intervals $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$. On each of these intervals f' has a constant sign. We choose a test point from each interval.

1. On $(-\infty, 0)$, $f'(-1) = 9 > 0$ and f is increasing
2. On $(0, 2)$, $f'(1) = -3 < 0$ and f is decreasing
3. On $(2, \infty)$, $f'(3) = 9 > 0$ and f is increasing

■

12.0.5 Exercise

Find the intervals of constant monotonicity for the function $f(x) = x^2 - x$.
■

12.1 Concavity

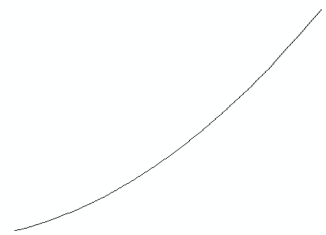
We can determine the monotonicity of f' by applying the above analysis and constructing a sign chart for f''. But what does the monotonicity of f' tell us about the graph of f. Quite a lot! There are essentially four different cases here, depending on whether f' is increasing or decreasing, and positive or negative. We consider each below:

12.1.1 Example

1. f' increasing and positive.

   Since f' is positive, f has positive sloped tangents.

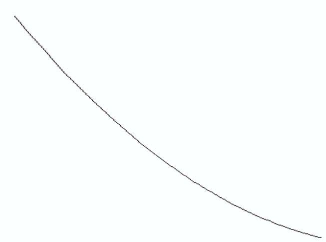
   Since f' is increasing, the tangents to f are getting steeper (larger positive).



2. f' increasing and negative.

   Since f' is negative, f has negative sloped tangents.

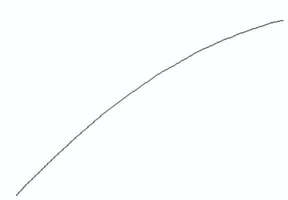
   Since f' is increasing, the tangents to f are getting flatter ("smaller" negative).



3. f' decreasing and positive.

   Since f' is positive, f has positive sloped tangents.

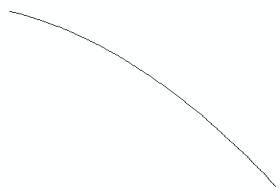
   Since f' is decreasing, the tangents to f are getting flatter (smaller positive).



4. f' decreasing and negative.

Since f' is negative, f has negative sloped tangents.

Since f' is decreasing, the tangents to f are getting steeper ("larger" negative).



■

In the first two of these cases, f is said to be *concave up*. In the second two f is *concave down*. A point at which f changes concavity is called an *inflection point*. To compute the concavity of f, we merely apply the monotonicity theorem to f' and construct a sign chart for f''.

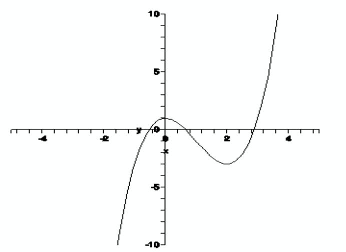
12.1.2 Example

For $f(x) = x^3 - 3x^2 + 1$ from our example above, the root of $f''(x) = 6x - 6$ is $x = 1$. The sign chart for f'' is as follows

1. $$\text{On } (-\infty, 1),\ \ f''(0) = -6 < 0 \text{ so } f \text{ is concave down}$$

2. $$\text{On } (1, \infty),\ \ f''(2) = 6 > 0 \text{ so } f \text{ is concave up}$$

Putting the concavity of f together with the monotonicity from the previous example, we get an idea of the shape and direction of the graph of f given below:



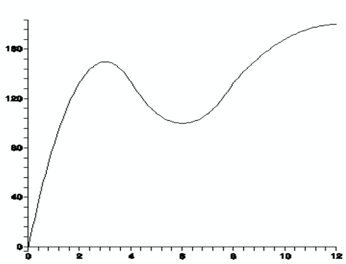
■

12.1.3 Exercise

Find the intervals of constant concavity and the inflection points for $f(x) = x^3 - 3x$. ■

12.1.4 Exercise

For each function below, find all intervals on which f is increasing, all intervals on which f is decreasing, all intervals on which f is concave up, all intervals on which f is concave down, and all inflection points of f.

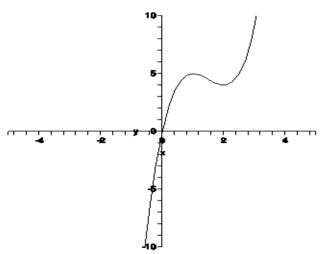
1. $f(x) = 5 - 4x - x^2$
2. $f(x) = 5 + 12x - x^3$
3. $f(x) = \sin x$ on the interval $[0, 2\pi]$
4. 

13 Local Extrema

A *local (or relative) extremum* is a point on a graph where the function value has a maximum (minimum) value when compared with other function values "nearby". For example, a local maximum does not have to possess the largest function value overall (this is called an *absolute* maximum), but merely the largest when compared with all other points in some open interval around the point. A simple example will illustrate this point.

13.0.5 Example

The graph of the function $f(x) = 2x^3 - 9x^2 + 12x$ is shown below.



f has a local maximum at $x = 1$ at the peak of the curve where the tangent line is horizontal. If we took an interval of width $\frac{1}{10}$ around $x = 1$, for example, then the function value $f(1)$ would be the largest in that interval.

Similarly, the function has a local minimum at $x = 2$, where the trough occurs.

Note that this function has *no absolute extrema*, as it grows without bound in both the positive and negative directions. ■

How do we find the local extrema? Recall that a sign chart for $f'(x)$ tells us the monotonicity of f. And clearly, if f changes monotonicity at a point, then there is a local extremum there. How can we identify such points? First, we need to find all points where this can *possibly* occur.

13.0.6 Definition

A *critical point* for a function $f(x)$ is a point $x = a$ in the domain of f where either

1. $f'(a) = 0$ (also called a stationary point)

2. $f'(a)$ does not exist.

■

It turns out that local extrema must occur at critical points, but not all critical points give rise to local extrema. So to find local extrema, we first find critical points, and then test them in the following way:

13.0.7 Example

1. In the example $f(x) = 2x^3 - 9x^2 + 12x$ given above, $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$. Thus we have that $f'(1) = f'(2) = 0$, so $x = 1$ and $x = 2$ are the critical points. How do we test whether they are local extrema? Recall that the sign chart for f' tells us where f is increasing or decreasing. If f changes direction (i.e. if f' changes sign), then we have a local extremum.

2. To see that not all critical points give rise to local extrema, consider the simple function $f(x) = x^3$. The root of $f'(x) = 3x^2 = 0$ is $x = 0$. So this is the only critical point. A sign chart shows that $f'(x) > 0$ for all $x \neq 0$, and thus f is increasing everywhere. Even though f has a horizontal tangent (stationary point) at $x = 0$, the graph doesn't turn over, but continues to increase there.

■

To identify the local extrema, we thus identify the sign changes of f' occuring at critical points. This is codified in the following theorem:

13.1 First Derivative Test

Suppose $x = a$ is a critical point for f. Then

1. If f' changes from positive to negative at $x = a$, then $f(a)$ is a local maximum.

2. If f' changes from negative to positive at $x = a$, then $f(a)$ is a local minimum.

3. If f' does not change sign at $x = a$, then $f(a)$ is not a local extremum.

■

13.1.1 Exercise

Find all critical points and then identify the local extrema:

1. $f(x) = x^2 - 1$
2. $f(x) = x^3 - 3x^2 + 3x - 1$

■

13.1.2 Example

Consider the function $f(x) = x^{\frac{1}{3}}$. Then $f'(x) = \dfrac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3x^{\frac{2}{3}}}$.

Notice that f' is never zero so that f has no stationary points. However, f does have a critical point at $x = 0$, since $x = 0$ is in the domain of f but $f'(0)$ is undefined (f has a vertical tangent at $x = 0$). To construct a sign chart for f', we use the critical point $x = 0$. But we see that f' is positive on $(-\infty, 0)$ and on $(0, \infty)$ so that f has no local extremum. ■

13.1.3 Exercise

Find the critical points and local extrema for the function $f(x) = x^{\frac{2}{3}}$. ■

There is another test to determine local extrema at stationary points. We observe that a function has a local maximum at a stationary point only if it is concave down there. Similarly, a local minimum at a stationary point occurs where the function is concave up. This is stated in the following:

13.2 Second Derivative Test

Suppose $x = a$ is a stationary point for f and f is differentiable twice at $x = a$. Then

1. If $f''(a) > 0$ then f has a relative minimum at $x = a$.

2. If $f''(a) < 0$ then f has a relative maximum at $x = a$.

3. If $f''(a) = 0$ then the test is inconclusive.

■

13.2.1 Example

In our original example $f(x) = 2x^3 - 9x^2 + 12x$, we have critical points at $x = 1$ and $x = 2$. We compute $f''(x) = 12x - 18$ and evaluate $f''(1) = -6 < 0$ and $f''(2) = 6 > 0$. Thus by the second derivative test, f has a local maximum at $x = 1$ and a local minimum at $x = 2$.

13.3 Graphing Polynomials

If we combine the notion of local extrema with monotonicity and concavity, we can get a fairly good idea of what the graph of a function looks like. Also, for polynomials we can determine the end behavior (i.e. $\lim_{x \to \pm\infty}$) and look for x and y intercepts.

13.3.1 Example

Consider $f(x) = x^3 + x^2 - x - 3$.

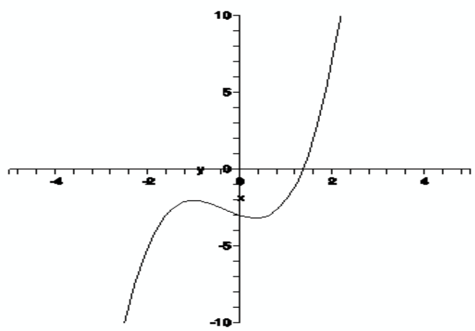
1. INTERCEPTS: We cannot analytically solve $f(x) = 0$, but the y-intercept is at $f(0) = -3$.

2. LOCAL EXTREMA: Computing $f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$ shows that f has critical points at $x = -1$ and $x = \frac{1}{3}$. Furthermore, a sign chart for $f'(x)$ and the first derivative test shows that f has a local maximum at $x = -1$ and a local minimum at $x = \frac{1}{3}$.

3. MONOTONICITY: The sign chart constructed above for $f'(x)$ shows that f is increasing on $(-\infty, -1)$ and on $(\frac{1}{3}, \infty)$ and is decreasing on $(-1, \frac{1}{3})$.

4. CONCAVITY: Computing $f''(x) = 6x+2$ and constructing a sign chart for f'' shows that f is concave down on $(-\infty, -\frac{1}{3})$ and concave up on $(-\frac{1}{3}, \infty)$. The point $x = -\frac{1}{3}$ is the only inflection point for f.

5. END BEHAVIOR: We note that $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3 = -\infty$ and that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^3 = \infty$.



■

14 Absolute Extrema

An *absolute extremum* is a maximum (or minimum) function value over an entire interval. There are typically three types of intervals we will consider: closed intervals, open intervals, and the entire real line.

14.1 Absolute Extrema on Closed Intervals

We first take up the case of closed intervals, where we can apply the following theorem:

14.1.1 Extreme Value Theorem

If f is continuous on the closed interval $[a, b]$, then f has both a maximum and minimum on $[a, b]$. Moreover, these extrema occur either at critical points interior to $[a, b]$ or at the endpoints a and b. ■

So to determine absolute extrema for a continuous function on the closed interval $[a, b]$, we first find any interior critical points. Then we compute the function values at these critical points, together with the function values at the endpoints. The largest and smallest of these values are the absolute extrema.

14.1.2 Example

Consider the function $f(x) = 2x^3 - 15x^2 + 36x$ on the closed interval $[1, 5]$. The critical points for f are the roots of $f'(x) = 6x^2 - 30x + 36$ which are $x = 2, 3$. Both of these critical points are interior to the interval $[1, 5]$. So we compute $f(1) = 23$, $f(2) = 28$, $f(3) = 27$, and $f(5) = 55$. The EVT then tells us that $f(1) = 23$ is the absolute minimum and $f(5) = 55$ is the absolute maximum. ■

14.1.3 Exercise

1. Find the absolute extrema for $f(x) = 8x - x^2$ on the interval $[1, 5]$.

2. Find the absolute extrema for $f(x) = \sin x$ on the interval $[0, \frac{5\pi}{4}]$.

■

14.2 Absolute Extrema on $\mathbb{R}$

Here things aren't always quite as simple as in the closed interval case. First we should look at the end-term behavior of our function. In other words, try to compute $\lim_{x\to\pm\infty} f(x)$. We could also look at any critical points/local extrema. Finally, to get a clear picture globally, we might construct or consult a graph of f.

14.2.1 Example

Consider the absolute extrema of $f(x) = \frac{1}{x}$ on $\mathbb{R}$. Here we have $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. In other words, f has a horizontal asymptote at $y = 0$. To find the critical points, note that $f'(x) = -\frac{1}{x^2}$ has no real roots and does not exist at $x = 0$. While $x = 0$ is technically not a critical point (f is not defined there), it is still a point of interest. Indeed, $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$ so that f can have no absolute extrema. ■

One class of functions are easy to handle in this case, and that is the polynomials. As we have seen, the endterm behavior of polynomials is completely determined by the leading coefficient and whether the degree is even or odd. In fact, any odd degree polynomial has no absolute extrema on $\mathbb{R}$ since the graph goes to opposite infinities in opposite directions. On the other hand, an even degree polynomial will always have exactly one absolute extremum, and it must occur at a critical point.

14.2.2 Example

To find the absolute extrema of $f(x) = 2x^4 + 8x$ on $\mathbb{R}$, we first note that

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2x^4 = \infty$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2x^4 = \infty$$

so that f has no absolute maximum. To find the absolute minimum, we find the critical points. Since $f'(x) = 8x^3+8$ has only the single real root $x = -1$, this must be where the absolute minimum occurs. To verify this, construct a sign chart for f' to see that this critical point is also a local minimum, and the only local extrema. ■

14.2.3 Exercise

Find the absolute extrema on $\mathbb{R}$ for the following polynomials:

1. $$f(x) = 3x^7 - 5x^2 + 8$$

2. $$f(x) = 2x^2 - 7x + 4$$

3. $$f(x) = -3x^4 + 4x^3$$

■

14.3 Absolute Extrema on Open Intervals (a, b)

This case is similar to the real line case shown above. As we mentioned above, if f does have an absolute extremum in this case, it must occur at a local extremum. We state this formally as:

14.3.1 Absolute Extrema on Open Intervals

Any absolute extremum on an open interval (or $\mathbb{R}$) must also be a local extremum. ■

When considering absolute extrema on $\mathbb{R}$, we were interested in the end-term behavior of f (i.e. $\lim\limits_{x\to\pm\infty} f(x)$). Analogously, when we consider absolute extrema on an open interval (a, b), we should look at the limits as x approaches the endpoints of the open interval. In other words, we should compute the limits

$$\lim_{x \to a^+} f(x)$$
$$\lim_{x \to b^-} f(x)$$

14.3.2 Example

1. Consider the function $f(x) = \dfrac{1}{x}$ on the interval $(2, 5)$. Since f has no critical points, there are no local extrema for f on the interval $(2, 5)$. Indeed, looking at a sign chart for $f'(x)$ shows that f is decreasing on the interval $(2, 5)$. Thus the logical candidates for absolute extrema would be the endpoints. However, we are only considering the open interval and the endpoints aren't included. Essentially, we are looking at the graph of f between 2 and 5 but with holes at the endpoints $x = 2$ and $x = 5$. Since the function values get arbitrarily close to $\dfrac{1}{2}$ from below and to $\dfrac{1}{5}$ from above, we see that f has no absolute extrema on $(2, 5)$.

2. Consider the function $f(x) = \dfrac{1}{x^2 - x}$ on the interval $(0, 1)$. We first note that f has vertical asymptotes at both endpoints $x = 0$ and $x = 1$. We then compute the limits

   $$\lim_{x \to 0^+} f(x) = -\infty$$
   $$\lim_{x \to 1^-} f(x) = -\infty$$

   Since f is continuous on the entire open interval $(0, 1)$ it follows that f must have an absolute maximum but no absolute minimum. If we compute f' we see that f has a critical point at $x = \dfrac{1}{2}$. Constructing the sign chart for f' shows that f has a local maximum at the critical point $x = \dfrac{1}{2}$. Thus, f has an absolute maximum at $x = \dfrac{1}{2}$ on the interval $(0, 1)$.

■

15 Optimization

15.1 Example

Find two positive numbers whose sum is 22 and whose product is a maximum.

1. Find two positive numbers, call them x and y.

2. The product is to be a maximum, so the the function to be maximized (the *objective function*) is

$$f(x) = xy$$

3. The sum of the two numbers must be 22, so that

$$x + y = 22$$

   This is called the *constraint equation*

4. Solution:

   (a) First solve the constraint for y, so that

$$y = 22 - x$$

   (b) Substitute into the objective function, so that

$$f(x) = xy = x(22 - x)$$

   (c) Optimize $f(x) = x(22 - x)$, considering that x (and y) must be positive, so the domain we are interested in is all $x > 0$, which is the interval $(0, \infty)$

   Since $f'(x) = 22 - 2x$, we have a single critical point at $x = 11$. Constructing a sign chart for f' shows that f has a local maximum at $x = 11$. Since this is the only local maximum, it must also be the absolute maximum (it's the vertex of the parabola!).

   Thus one of the two numbers is $x = 11$. To find the other number, we return to the constraint equation, so that $y = 22 - 11 = 11$ also. So the two numbers are 11 and 11, their sum is 22, and their product, 121, is the desired maximum.

15.2 How To Solve Optimization Problems

GOAL

Optimize (either maximize, minimize, or both) a particular function on a particular interval.

HINTS

1. Read the problem carefully
2. Identify what is being asked for. Usually, these are your *variables*.
3. Identify what is to be optimized. Write this function (call it the *objective function*) in terms of your variables.
4. Identify any *constraint equations* on your variables. Use these to write one variable as a function of the other.
5. Since the objective function is likely a function of 2 (or more) variables, use the constraint equation(s) to reduce your objective function to one of a single variable (do this by solving the constraint equation for one variable and substituting into the objective function).

SOLUTION

Now optimize your objective function. You can use the first derivative test to identify any local extrema. Be sure to consider any restrictions on the domain of your objective function. In other words, what is the range of values your variables can take? (e.g. are they positive, nonzero, etc). Finally, apply all you know about finding extrema to determine the absolute maximum (or minimum) of your objective function. This value is the optimal value of the objective function. Where it occurs (i.e. for what value of the variable) is the value of your variable that optimizes the objective function. If necessary, go back to the constraint equation to solve for the other variable.

15.3 Exercise

A woman is on a beach 6 miles due south of a lighthouse on the beach that is directly 3 miles due east of an island. It is 2:00 p.m. on a Saturday. The woman must get to the island by 3:30 in order to attend a wedding. To reach the island, she can walk along the beach (toting her inflatable boat), and then put her boat into the water and row the rest of the way, directly to the island. She can row the boat at 4 miles per hour and she can walk at 5 miles per hour. Find the point along the beach where she should launch the boat so that her path to the island is taken in the minimum time. Will she make it in time for the wedding if she takes the path of minimum time?

1. "Find the point along the beach"...., or find the distance she should walk from her current position until she launches the boat. Call that distance (in miles) x.

2. "Find the point along the beach....so that her path to the island is taken in the minimum time", or minimize the time. Call the time function T. If she walks x miles at 5 miles an hour, then that part of the trip will take her $\frac{x}{5}$ hours. What about the remainder of the trip?

3. If she walks x miles up the beach, what is the remaining distance she must row (call it y)? Well, there would be $6-x$ miles remaining along the beach to the lighthouse, and 3 miles from the lighthouse to the beach, so her path in the water would constitute the hypotenuse of a right triangle with sides of length $6-x$ and 3. Thus, we have the constraint that

$$y^2 = 3^2 + (6-x)^2$$

or that

$$y = \sqrt{9 + (6-x)^2}$$

(we can disregard the negative square root since y is a distance). Draw a picture!!!!! Finally, if her distance in the water is $y = \sqrt{9 + (6-x)^2}$ miles, then the time it will take her to row that distance at 4 miles per hour is

$$\frac{y}{4} = \frac{\sqrt{9 + (6-x)^2}}{4}$$

hours.

4. Substitute the constraint into the objective function, so that time $T =$ time to walk + time to row, or

$$T(x) = \frac{x}{5} + \frac{y}{4} = \frac{x}{5} + \frac{\sqrt{9 + (6 - x)^2}}{4}$$

5. Now use the calculus to minimize T. Don't forget that x is non-negative. Also, you might assume that x is not greater than 6, since if that were the case, she would have walked past the lighthouse and would then have to row back south-west. Although, it would be interesting to see if that is a valid assumption. Does her time continue to increase as x becomes greater than 6? It should! Identify the optimal path and the minimal time. Will she make it to the wedding?

6. For fun, try the problem again, this time assuming she is toting along a small motor for her boat, so that her speed in the water would increase to 10 miles per hour, but her walking speed would decrease to 3 miles per hour (she's lugging a small motor!). Before you begin, think about what the *logical* solution in this case might be.

■

16 Mean Value Theorem

A very useful tool is the Mean Value Theorem for derivatives. It has a very nice graphical interpretation. It is usually easier to see if we first investigate a special case, called

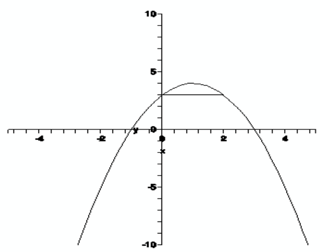
16.1 Rolle's Theorem

Suppose f is differentiable on (a, b) and continuous on $[a, b]$. Furthermore, suppose that $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ with $f'(c) = 0$.

■

16.1.1 Example

To see the graphical interpretation of Rolle's Theorem, we consider the graph below of the function $f(x) = -x^2 + 2x + 3$ on the interval $[0, 2]$.



Notice that f is differentiable everywhere and that $f(0) = f(2) = 3$ so that the hypotheses of Rolle's Theorem are satisfied. The conclusion is then that there must be some point $c \in (0, 2)$ with $f'(c) = 0$. Can you find the point c?

■

Notice that Rolle's theorem says that if you start at $x = a$ and trace differentiably to $x = b$, then since f has the same function value at a and b, the graph must turn over somewhere in between (at the point c). But "turning over" differentiably means precisely that there is a horizontal tangent line.

16.2 Mean Value Theorem

The Mean Value Theorem (MVT) is a simple generalization of Rolle's Theorem. We don't require $f(a) = f(b)$, but construct the secant line through $x = a$ and $x = b$. The MVT then concludes the existence of a tangent line to f that is parallel to this secant line. If we "tilt our heads" to the appropriate angle to make the secant line horizontal, then we are looking at Rolle's Theorem.

16.2.1 Mean Value Theorem

Suppose f is differentiable on (a, b) and continuous on $[a, b]$. Then there exists a point $c \in (a, b)$ with

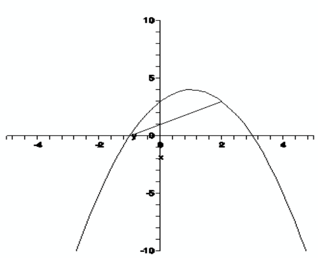
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

■

16.2.2 Example

Using the function $f(x) = -x^2 + 2x + 3$ as in the above example but on the interval $[-1, 2]$, we see that the slope of the secant line is

$$\frac{f(2) - f(-1)}{2 - (-1)} = 1$$



Thus the MVT says there is some point c in the interval $[-1, 2]$ with $f'(c) = 1$. To find this point, we solve $f'(x) = -2x + 2 = 1$ for x to find $c = \frac{1}{2}$. ■

16.2.3 Exercise

Verify that the hypotheses of the MVT are satisfied and find all points c guaranteed by the theorem for the function $f(x) = x - \frac{1}{x}$ on the interval $[3, 4]$. ■

17 Area Under a Curve

17.1 Approximating Area with Rectangles

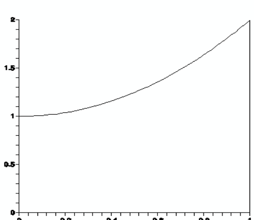
The ancient Greeks spent lots of time developing formulas for the area of various plane geometric figures with *linear* sides. In other words; triangles, squares and other various polygons. They also were familiar with the area of a circle. But the problem of computing the area under any general *non-linear curve* was not solved until the calculus came along. Indeed, this area problem was the impetus for the development of the Integral Calculus.

Problem: Find the area below the graph of a non-linear function $f(x)$ and above the x-axis on the interval $[a, b]$ (where $f(x) \geq 0$ on $[a, b]$)

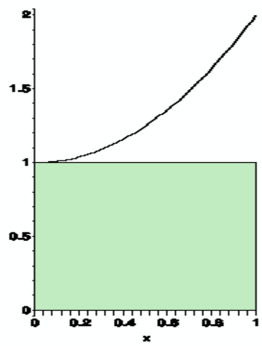
First attempt at solution: Approximate the area using areas that the Greeks knew. For starters, let's use rectangles. The example below discusses this method in more detail.

17.1.1 Example

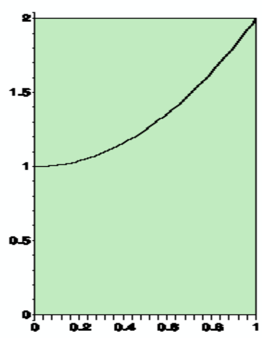
We wish to approximate the area under the graph of $f(x) = x^2 + 1$ (and above the x-axis) on the interval $[0, 1]$ as shown by the enclosed region in the graph below:



We will first approximate this area using a single rectangle as shown below.



The rectangle has width 1 (it lies on the interval $[0, 1]$) and has height $f(0) = 0^2 + 1 = 1$. So the area of the rectangle is $1 \times 1 = 1$. Since we chose the height of our rectangle at $x = 0$ the left endpoint of the interval $[0, 1]$, this is called a "left-endpoint rectangle" or more concisely a *left rectangle*. We could also make an approximation using the corresponding *right rectangle* as shown below.

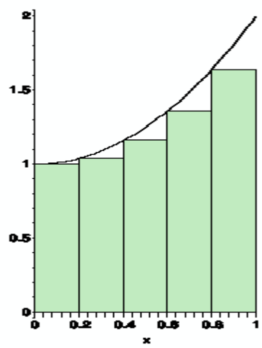


Here, the rectangle has width one and height equal to $f(1) = 1^2 + 1 = 2$. So the area of the rectangle is $1 \times 2 = 2$.

Notice that the left rectangle area of one is too small of an approximation since it lies entirely below the graph of f. Similarly, the right rectangle area of 2 is too big since it lies above the graph of f.

It is thus interesting to note that with these two simple approximations, we can say with certainty that the area under the curve is between 1 and 2.

But certainly we can do better. How? Use more rectangles!! To do this we first subdivide the interval $[0, 1]$ into subintervals. We then construct left (or right) rectangles on each subinterval. The picture below shows five left rectangles.



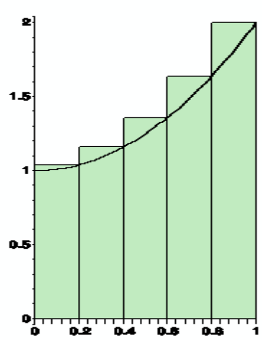
Now to compute our area approximation we simply add up the area of the five rectangles. Each of these has width $\frac{1}{5}$ and the heights are determined by the function value at the left endpoint of each subinterval. Thus the sum of the five rectangle areas is

$$\frac{1}{5} \times f(0) + \frac{1}{5} \times f(.2) + \frac{1}{5} \times f(.4) + \frac{1}{5} \times f(.6) + \frac{1}{5} \times f(.8)$$

$$= \frac{1}{5}[f(0) + f(.2) + f(.4) + f(.6) + f(.8)]$$

$$= \frac{1}{5}[(0^2 + 1) + (.2^2 + 1) + (.4^2 + 1) + (.6^2 + 1) + (.8^2 + 1)] = 1.24$$

Once again we can say that this estimate of 1.24 is too small since all of the rectangles lie entirely below the curve. We then can approximate using five right rectangles as shown below.



The sum of the five right rectangles is shown below. I've spared you the details of the arithmetic.

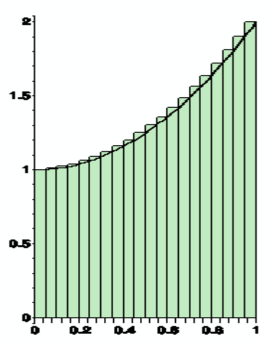
$$\frac{1}{5} \times f(.2) + \frac{1}{5} \times f(.4) + \frac{1}{5} \times f(.6) + \frac{1}{5} \times f(.8) + \frac{1}{5} \times f(1) = 1.44$$

So now we have an approximation of 1.44 that we know is too big since the five rectangles are above the curve. Putting this together with the five left rectangles we see that the area under the curve must be between 1.24 and 1.44.

Again, this is not too bad of an estimate for what little work we had to do. So the next logical step is to do this with more rectangles. As a general rule of thumb:

As the number of rectangles increases, the approximation becomes better

However, as the number of rectangles increases, the number of calculations increases and the computation takes longer. Below is the picture for 20 right rectangles. One can see that while this estimate will still be too big, it will be more accurate.



Using Maple I computed the area of the 20 right rectangles as 1.35875. Similarly, I computed the area of the 20 left rectangles as 1.30875. We've now narrowed it down significantly. We know for certain that the actual area begins with 1.3........!!!!!!

I can't resist. Let's do 1000 left and right rectangles. Using Maple again, this gives us the area A bounded as

$$1.3328335 < A < 1.3338335$$

Of course, 1000 is a relatively tiny number of rectangles, but you get the idea. Finally, this is all really nice, but it's still only an approximation. To get the *actual* area, we mimic the above procedure and then take a

limit as the number of rectangles goes to infinity

We will return to this notion later.

17.2 Relationship of Area to Derivatives

17.2.1 Example

Consider the area under the graph of the function $f(x) = 2x + 1$ on the interval $[1, x]$ (where $x > 1$). We can actually calculate the area *exactly* using the Greek technique.

If we think of the region as a triangle on top of a rectangle, we see that

Area of Rectangle = Height $\times$ Width = $3 \times (x - 1)$.

Area of Triangle = $\frac{1}{2}$ Base $\times$ Height =$\frac{1}{2}(x - 1) \times [(2x + 1) - 3]$.

Thus the total area under the curve is

$$A(x) = 3(x - 1) + \frac{1}{2}(x - 1) \times (2x - 2)$$

$$A(x) = x^2 + x - 2$$

In particular, we see that $A(1) = 0$ as it should.

So what does this have to do with the derivative? Notice that the derivative function for $A(x)$ is given by $A'(x) = 2x + 1$. But this is the original function $f(x)$ under which we were calculating the area!!

$$A'(x) = f(x)$$

So it seems that the derivative of the area function $A(x)$ is the function $f(x)$ that we are trying to compute the area under.

If we start with the function $f(x)$, then to find the area function, we need to find a function whose derivative is $f(x)$. This is the reverse of the process of differentiation and is called **antidifferentiation**. This leads us to take a brief respite from the area problem and study antiderivatives in more detail.

18 Antiderivatives

The process of *antidifferentiation* is essentially the reverse of the process of differentiation. Instead of starting with a function f and differentiating it, we start with a function f and find a function whose derivative is f. More precisely:

18.1 Definition of Antiderivative

An *antiderivative* (AD) for a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. ■

18.1.1 Example

1. We know that the derivative of $F(x) = x^2$ is the function $f(x) = 2x$. Thus F is an antiderivative of f. But there are others. How about $x^2 + 3$, or $x^2 - 7$?

2. Starting with $f(x) = x^3 - 7x$, by doing some intelligent guessing, we find an antiderivative $F(x) = \frac{1}{4}x^4 - \frac{7}{2}x^2$ ■

Note that in the first example above, we see that antiderivatives are by no means unique. Indeed, once you have found one AD F, there are infinitely many others found by adding arbitrary constants to F. We use the following notation and terminology to express this idea:

18.1.2 Definition of Indefinite Integral

The *indefinite integral* of a function $f(x)$ is the collection of all antiderivatives for f and is denoted by

$$\int f(x)\ dx$$

If F is one AD for f, then to represent the entire class of AD's we write

$$\int f(x)\ dx = F(x) + C$$

where C denotes an arbitrary constant.

18.1.3 Exercise

Find

$$\int \sin x \; dx$$

■

The *intelligent* guessing we did in the example above was really just reversing the process of the power rule. Similarly, in the exercise above we found the AD's for $\sin x$ by recalling the derivatives of the trig functions and working backwards. We can formalize this with a list of integrals:

18.2 Integration Rules

1. Anti-power rule

$$\int x^n \; dx = \frac{x^{n+1}}{n+1} + C \qquad (\text{for} \;\; n \neq -1)$$

2.

$$\int \sin x \; dx = -\cos x + C$$

3.

$$\int \cos x \; dx = \sin x + C$$

4.

$$\int \sec^2 x \; dx = \tan x + C$$

5.

$$\int \sec x \tan x \; dx = \sec x + C$$

6.

$$\int \csc^2 x \; dx = -\cot x + C$$

7.

$$\int \csc x \cot x \; dx = -\csc x + C$$

The linearity properties for derivatives have analogs in the antiderivative case:

18.3 Linearity Rules

1. $$\int kf(x)\ dx = k\int f(x)\ dx \quad \text{for constants } k$$

2. $$\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

■

Using the linearity rules together with the anti-power rule and trig integrals, we are able to antidifferentiate any polynomial and other similar functions.

18.3.1 Exercise

1. $$\int (x^3 - 7x^2 + 12x - 6)\ dx$$

2. $$\int (4\cos x + 7\sec^2 x)\ dx$$

3. $$\int (\sqrt{x} - x^{\frac{3}{4}} - \sec x \tan x)\ dx$$

Unfortunately, we don't have the tools to integrate many other functions at this point. In fact, we will spend a couple of months in Calculus 2 doing just that. However, we do study one technique in the next section; Integration by Substitution.

19 Integration by U-Substitution

Recall that the chain rule for differentiating a composition of two functions is

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x)$$

Reversing this idea allows us to integrate various functions. To this end, we see that $F(x) = f(g(x))$ is an antiderivative for the function $f'(g(x)) \cdot g'(x)$, or, in other words

$$\int f'(g(x)) \cdot g'(x)\ dx = f(g(x)) + C$$

The key to using this fact is to recognize when an integrand is of the form above. Here is an example explaining the notation.

19.0.2 Example

To compute

$$\int 2x\cos(x^2)\ dx$$

we note that $g(x) = x^2$ gives $g'(x) = 2x$. So the integrand is of the proper form, with $f'(x) = \cos x$. Thus, $f(x) = \sin x$ and we have

$$\int 2x\cos(x^2)\ dx = \int f'(g(x)) \cdot g'(x)\ dx = f(g(x)) + C = \sin(x^2) + C$$

To check this result, we use the chain rule to compute

$$\frac{d}{dx}(\sin(x^2) + C) = \cos(x^2) \cdot 2x$$

■

An alternative and more compact notation involves letting $u = g(x)$ so that $du = g'(x)dx$. This gives an alternate version, and is the impetus for the terminology ”u-substitution”.

$$\int f'(u)\ du = f(u) + C$$

The key in applying this form is to recognize in the integrand some ”inside” function u, whose derivative du sits ”outside”. We can then make a formal change of variables to take care of the chain rule issue. The following example illustrates this technique.

19.0.3 Example

1. Consider the integral from the previous example, $\int 2x\cos(x^2)\ dx$.

   Here we notice the "inside" function is $u = x^2$ and its derivative $du = 2xdx$ sits outside. Making this formal substitution gives

   $$\int 2x\cos(x^2)\ dx = \int \cos u\ du = \sin u + C = \sin(x^2) + C$$

2. To compute $\int x(x^2-1)^{28}\ dx$, we notice the "inside" function is $x^2 - 1$. Its derivative, $2x$, doesn't sit "outside" exactly, but a multiple of it does. We can see that we don't need the actual derivative "outside", but any constant multiple of it will do. So we let

   $$u = x^2 - 1$$

   $$du = 2x\ dx$$
   $$\frac{1}{2}du = x\ dx$$

   Thus making the appropriate substitutions, we get

   $$\int x(x^2-1)^{28}\ dx = \int \frac{1}{2}u^{28}du$$

   The integral with respect to u can easily be computed and then converted back to the appropriate function of x.

   $$\int x(x^2-1)^{28}\ dx = \int \frac{1}{2}u^{28}du = \frac{1}{2}\frac{u^{29}}{29} + C = \frac{1}{58}(x^2-1)^{29} + C$$

   Again, its not a bad idea to check our result. To do this, differentiate our antiderivative using the chain rule to see that

   $$\frac{d}{dx}\Big(\frac{1}{58}(x^2-1)^{29} + C\Big) = \frac{1}{58}\cdot 29(x^2-1)^{28}\cdot 2x = x(x^2-1)^{28}$$

■

19.0.4 Exercise

1. $$\int \cos \frac{\pi}{3}x \ dx$$

2. $$\int x^3\sqrt{5+x^4} \ dx$$

3. $$\int [\csc(\sin x)]^2 \cos x \ dx$$

4. $$\int \frac{5x^4}{(x^5+1)^3} \ dx$$

5. $$\int \frac{\sin(\frac{1}{x})}{3x^2} \ dx$$

20 Area and Riemann sums

Assume that $f(x) \geq 0$ on $[a, b]$ and denote the area under f on $[a, b]$ by A.

20.1 Partition $[a, b]$

We first subdivide the interval $[a, b]$ into n equal subintervals of width

$$\Delta x = \frac{b-a}{n}$$

The endpoints of the subintervals are denoted by $x_0, x_1, x_2, \ldots x_{n-1}, x_n$. Since we add Δx to each endpoint to get to the next endpoint, we have that

$$x_0 = a \quad x_1 = a + \Delta x \quad x_2 = a + 2\Delta x \quad x_3 = a + 3\Delta x \quad \ldots$$

So that $\mathbf{x_k = a + k\Delta x}$. Notice then that

$$x_n = a + n\Delta x = a + n\left(\frac{b-a}{n}\right) = a + (b-a) = b$$

20.2 Compute the sum of the n rectangle areas

Let x_k^* denote any point in the k^{th} subinterval $[x_{k-1}, x_k]$, so that $f(x_k^*)$ denotes the height of the k^{th} rectangle taken at the point $x_k^* \in [x_{k-1}, x_k]$. For example:

Using right endpoint rectangles, $x_k^* = x_k$

Using left endpoint rectangles, $x_k^* = x_{k-1}$

Using midpoint rectangles, $x_k^* = \frac{1}{2}(x_{k-1} + x_k)$

Then the area of the k^{th} rectangle is height $\times$ width=$f(x_k^*)\Delta x$. Summing up these n rectangle areas we have that the total area under f on $[a, b]$ is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Any sum of this form is called a **Riemann sum**. Notice that this sum is an **approximation** to the area under f, and that this approximation gets better as n increases. To get the actual area A, we take the limit of the Riemann sum as n goes to infinity. In other words, we **define**

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Furthermore, if this limit exists (independent of the partitioning we use for $[a,b]$), then f is said to be **integrable** on $[a,b]$ and we denote this limit by

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)\ dx$$

The last term in the above equation is called the **definite integral** of f on $[a,b]$.

20.2.1 Example

Compute the area under $f(x) = x^2 + 1$ on the interval $[1,4]$ using right endpoint rectangles

Partition $[1,4]$

Here we have that

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

Thus the partition for $[1,4]$ into n equal subintervals is given by

$$x_0 = a = 1 \quad x_1 = a + \Delta x = 1 + \frac{3}{n} \quad x_2 = 1 + 2\Delta x = 1 + 2(\frac{3}{n})\cdots$$

So that

$$x_k = a + k\Delta a = 1 + k(\frac{3}{n}) = 1 + \frac{3k}{n}$$

Compute the sum of n rectangle areas

Since we are using right endpoint rectangles, we have that $x_k^* = x_k$ so that the area of the rectangle on the k^{th} subinterval $[x_{k-1}, x_k]$ is given by

$$f(x_k^*)\Delta x = f(x_k)\Delta x = f(1 + \frac{3k}{n})(\frac{3}{n}) = \left((1 + \frac{3k}{n})^2 + 1\right)(\frac{3}{n})$$

$$= \left(1 + \frac{6k}{n} + \frac{9k^2}{n^2} + 1\right)(\frac{3}{n}) = \frac{6}{n} + \frac{18k}{n^2} + \frac{27k^2}{n^3}$$

We then form the Riemann sum to add the areas of all n rectangles:

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} \left(\frac{6}{n} + \frac{18k}{n^2} + \frac{27k^2}{n^3}\right)$$

Simplify the Riemann sum

Using the linearity of sums and the special sums from Theorem 5.4.2, we can rewrite the Riemann sum above in **closed form**, that is we can write it without the Σ and as a function of n.

$$\sum_{k=1}^{n} \left(\frac{6}{n}+\frac{18k}{n^2}+\frac{27k^2}{n^3}\right) = \sum_{k=1}^{n} \frac{6}{n}+\sum_{k=1}^{n} \frac{18k}{n^2}+\sum_{k=1}^{n} \frac{27k^2}{n^3} = \frac{6}{n}\sum_{k=1}^{n} 1+\frac{18}{n^2}\sum_{k=1}^{n} k+\frac{27}{n^3}\sum_{k=1}^{n} k^2$$

$$= \left(\frac{6}{n}\right)n + \left(\frac{18}{n^2}\right)\frac{n(n+1)}{2} + \left(\frac{27}{n^3}\right)\frac{n(n+1)(2n+1)}{6}$$

Simplifying, we have the sum of the n rectangle areas given in closed form as

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = 6 + \frac{9(n+1)}{n} + \frac{9(n+1)(2n+1)}{2n^2}$$

For example, if we want to compute the approximation to A using 2 right rectangles, we merely let $n = 2$ in the above expression to get $A \approx 6 + \frac{27}{2} + \frac{135}{8} = 36.375$. Of course this approximation is way too big (since the right rectangles circumscribe the curve). If we evaluate the above expression for $n = 1000$ right rectangles we get $A \approx 6 + \frac{9 \cdot 1001}{1000} + \frac{9}{2}\frac{1001 \cdot 2001}{1000^2} \approx 24.023$ which is a much better (but still too big) approximation to A. Finally, to get the **actual** value of A, we take the limit as n goes to infinity.

Calculate the actual area A

We defined the area under f on $[a, b]$ to be

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

We have already done the hard part by simplifying the Riemann sum. Using this simplified version and the linearity of limits we have:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty} \left(6 + \frac{9(n+1)}{n} + \frac{9}{2}\frac{(n+1)(2n+1)}{n^2} \right)$$

$$= \lim_{n\to\infty} 6 + \lim_{n\to\infty} \frac{9(n+1)}{n} + \lim_{n\to\infty} \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}$$

$$= 6 + 9 \lim_{n\to\infty} \frac{(n+1)}{n} + \frac{9}{2} \lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} = 6 + 9 \cdot 1 + \frac{9}{2} \cdot 2 = 24$$

20.3 Riemann sums as net signed area

Suppose that f is not necessarily positive on $[a, b]$, so that the Riemann sum does not represent an area under f on $[a, b]$. What does the Riemann sum represent in this case?

20.3.1 Example

Consider $f(x) = x^2 - 1$ on $[0, 1]$. Then the graph of f is entirely underneath the x-axis on this interval. Thus, the terms $f(x_k^*)$ do not represent rectangle heights, but are actually the **negative** of these heights. Thus the corresponding Riemann sum represents the sum of "negative" height rectangle areas. If we go through the calculations and compute the limit of the Riemann sum as in the above example, we see that this limit is $-\frac{2}{3}$. What this means is that the area **above** f and **below** the x-axis on the interval $[0, 1]$ is $\frac{2}{3}$.

20.3.2 Example

Consider the function $f(x) = \sin x$ on the interval $[0, 2\pi]$. On the first half of this interval, $[0, \pi]$, f is above the x-axis, while on the second half of the interval, $[\pi, 2\pi]$, f is below the x-axis. Furthermore, these two "areas" are the same (due to the symmetry of the sin function). Thus half of the rectangle areas are positive, while the other half are "negative". These two pieces of the Riemann sum will cancel each other out, so that

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = 0$$

21 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTOC) is just that: a theorem that fundamentally ties together both the integral and differential calculus. We give an informal proof of the FTOC below.

21.1 Proof of FTOC

Suppose that f is continuous on the closed interval $[a, b]$ and let F be an antiderivative for f. Partition $[a, b]$ into n equal subintervals of width Δx and with endpoints $a = x_0, x_1, \ldots x_{n-1}, x_n = b$. Since F is differentiable on $[a, b]$, it satisfies the hypotheses of the MVT for derivatives. On each subinterval $[x_{i-1}, x_i]$, we apply the MVT so that there exists a point $x_i^* \in [x_{i-1}, x_i]$ with

$$F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}$$

Noting that $F' = f$ and multiplying by Δx gives

$$f(x_i^*)\Delta x = F(x_i) - F(x_{i-1})$$

Now summing over all n subintervals gives

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + \cdots + [F(x_{n-1}) - F(x_{n-2})] + [F(x_n) - F(x_{n-1})]$$

But notice that this last sum is telescoping and all interior terms cancel leaving

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = F(x_n) - F(x_0) = F(b) - F(a)$$

Recall that the definite integral of f on $[a, b]$ is defined as the limit of the sum above, and we get:

21.2 FTOC

If f is continuous on $[a, b]$ and F is an antiderivative for f, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Alternatively, we use the notation $\Big[F(x)\Big]_a^b$ to denote $F(b) - F(a)$.

The beauty of the FTOC is the connection between the definite integral (i.e. the area problem) and the (anti-) derivative (i.e. the tangent line problem). It also allows for relatively easy computation of definite integrals and areas as shown in the next example(s).

21.2.1 Example

1. Since $F(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$ is an antiderivative for $f(x) = x^3 - x$, we have

$$\int_0^2 (x^3 - x)\ dx = \Big[\frac{1}{4}x^4 - \frac{1}{2}x^2\Big]_0^2$$

$$= (\frac{1}{4}2^4 - \frac{1}{2}2^2) - (\frac{1}{4}0^4 - \frac{1}{2}0^2) = 2$$

2. To find the area A under the graph of $f(x) = \sin x$ on the interval $[0, \pi]$, we note that an antiderivative is $-\cos x$ so that

$$A = \int_0^\pi \sin x\ dx = \Big[-\cos x\Big]_0^\pi$$

$$= (-\cos\pi) - (-\cos 0) = (--1) - (-1) = 2$$

■

21.2.2 Exercise

1. Find the area under the graph of $f(x) = \dfrac{1}{x^{\frac{2}{3}}}$ on the interval $[1, 27]$.

2. Compute $\displaystyle\int_0^{\frac{\pi}{4}} \left(\frac{1}{x^6} - \sec^2 x\right)\ dx$

■